GROMOV COMPACTNESS THEOREM FOR STABLE CURVES

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0. Introduction

The goal of this paper is to give a proof of the Gromov compactness theorem using the language of stable curves in the general situation, setting minimal assumption on almost complex structures and on pseudoholomorphic curves. In particular, we suppose that the almost complex structures on a target manifold are only *continuous* (*i.e.* of class C^0) and can vary.

More precisely, we consider a sequence $\{J_n\}$ of continuous almost complex structures on a manifold X which converges uniformly to a continuous structure J_∞. Furthermore, let $\{C_n\}$ be a sequence of Riemann surfaces with boundaries of fixed topological type. This means that all C_n can be parametrized by the same real surface Σ (see § 1 for details). Denote by $\delta_n : \Sigma \to C_n$ some parametrizations. Finally, let some sequence of J_n-holomorphic maps $u_n : C_n \to X$ be given.

Theorem 1. *If the areas of $u_n(C_n)$ are uniformly bounded (with respect to some fixed Riemannian metric on X) and the structures j_{C_n} of the curves C_n do not degenerate at the boundary (see* Definition 1.7*), then there exists a subsequence, still denoted (C_n, u_n), such that*

1) C_n converge to some nodal curve C_∞ in an appropriate completion of the moduli space of Riemann surfaces of given topological type, i.e. *there exist a parametrization map $\sigma_\infty : \Sigma \to C_\infty$ by the same real surface Σ;*

2) one can chose new parametrizations σ_n of C_n in such a way that each σ_n coincides with the given parametrization $\delta_n : \Sigma \to C_n$ outside some fixed compact subset $K \Subset \Sigma$ and the structures $\sigma_n^* j_{C_n}$ converge to $\sigma_\infty^* j_{C_\infty}$ in the C^∞-topology on compact subsets outside of the finite set of circles on Σ, which are pre-images of the nodal points of C_∞ by σ_∞;

3) maps $u_n \circ \sigma_n$ converge, in the C^0-topology on the whole Σ and in the $L^{1,p}_{\mathrm{loc}}$-topology (for all $p < \infty$) outside of the pre-images of the nodes of C_∞, to map $u_\infty \circ \sigma_\infty$, such that u_∞ is a J_∞-holomorphic map $C_\infty \to X$.

For the definitions involved and the formal statement we refer to § 1 and *Theorem 1.1.* Note, that this description of the convergence is precisely the one given by Gromov in [G]. Our notion of a *stable nodal curve* coincides, in fact, with the notion of a *cusp-curve* of Gromov, and with the notion of a *stable map* of Kontsevich and Manin [K-M]. The choice of terminology is explained by the fact that we prefer to consider our objects as curves rather than maps.

Theorem 1 generalizes the original result of Gromov in two directions. First, we note that the Gromov compactness theorem is still valid for continuous and continuously varying almost complex structures. This could have an interesting applications, since now one can consider C^0-small perturbations of an almost complex structure on a manifold being insured that at least compactness theorem still holds true.

Second, we consider not only the case of closed curves, but also the case when C_n are open and of a fixed "topological type", so that the complex structures of C_n can vary arbitrarily. In § 2 we study moduli spaces of open nodal curves. In particular, we define a natural complex structure for such moduli spaces and show that the condition of non-degeneration of complex structures of C_n near boundary is equivalent to boundedness of C_n in an appropriate completion of the moduli space.

Let us stop on this point, which is of independent interest. Fix a real oriented surface Σ of genus g with m marked points and with boundary consisting from b circles. Assume that $2g+m+b \geqslant 3$. Mark additionally a point on each boundary circle. Two complex structures J_1 and J_2 on Σ are isomorphic if there exists a biholomorphism $\varphi : (\Sigma, J_1) \to (\Sigma, J_2)$ isotopic to identity and preserving the marked points.

Topological space of complex structures modulo this equivalence relation will be denoted by $\mathbb{T}_\Sigma$, more precise description and notations are given in § 2.

For a non-closed Riemann surface $C = (\Sigma, J)$, we construct the holomorphic double of C which is a closed Riemann surface C^d containing C, and the holomorphic involution τ of C^d interchanching C with $\tau(C)$. Put $D^d = D + \tau(D)$, where D is the divisor of marked points, including boundary ones. We prove the following

Theorem 2. *There is a natural structure of complex manifold on $\mathbb{T}_\Sigma$ of complex dimension $3g-3+m+2b$ with tangent space $T_C\mathbb{T}\Sigma$ at C naturally isomorphic to the space $\mathsf{H}^1(C^d, \mathcal{O}(TC^d)\otimes\mathcal{O}(-D^d))^\tau$ of τ-invariants elements of $\mathsf{H}^1(C^d, \mathcal{O}(TC^d)\otimes \mathcal{O}(-D^d))$.*

Another result of this paper, which we would like to mention in the introduction, is an apriori estimate for pseudoholomorphic maps of "long cylinders", see *Second Apriori Estimate* in § 3. This estimate gives possibility to treat the degeneration of complex structure on the curves C_n and the "bubbling" phenomenon in a uniform framework of "long cylinders" and to get a precise description of the convergence near "neck" singularities where the usual "strong" convergence fails. In particular, this implies the Hausdorff convergence of the curves C_n in *Theorem 1*.

As an application the *Second Apriori Estimate*, we prove in *Corollary 3.6* the following generalization of removability theorem for the point singularity.

Theorem 3. *If the area of the image of J-holomorphic map $u : (\check{\Delta}, J_{\mathrm{st}}) \to (X, J)$ from the punctured disk into a compact almost complex manifold has "slow growth" ("is not growing too fast"), i.e. if $\mathsf{area}\,(u(R_k)) \leqslant \varepsilon$ for all annuli $R_k := \{z \in \mathbb{C} : \frac{1}{e^{k+1}} \leqslant |z| \leqslant \frac{1}{e^k}\}$ with $k >> 1$, then u extends to origin.*

The positive constant ε here depends only on the Hermitian structure (J, h) of X. This theorem under stronger assumption $\sum_k \mathsf{area}\,(u(R_k)) \equiv \|du\|^2_{L^2(\check{\Delta})} < \infty$ was proved by Sacks and Uhlenbeck [S-U] for harmonic maps, and by Gromov [G] for J-holomorphic maps. This fact (which is proved here for continuous J's) is new even in the integrable case. In fact, it measures the "degree of non-hyperbolicity" (in the sense of Kobayashi) of (X, J, h).

After the "inner" case considered in *Theorem 1*, we prove in § 5 the compactness theorem for curves with boundary on totally real submanifolds. For this "boundary" case we give appropriate generalizations of all "inner" constructions and estimates.

In particular, in *Corollary 5.7* we obtain a generalization of the Gromov's result about removability of boundary point singularity, see [G]. An improvement is that the statement remains valid also when one has *different* boundary conditions to the left and to the right from a singular point. Let us explain this in more details.

Define the (punctured) half-disk by setting $\Delta^+ := \{z \in \Delta : \mathsf{Im}(z) > 0\}$ and $\check\Delta^+ := \Delta^+ \backslash \{0\}$. Define $I_- :=]-1, 0[\subset \partial\check\Delta^+$ and $I_+ :=]0, +1[\subset \partial\check\Delta^+$. Let a J-holomorphic map $u : (\check\Delta^+, J_{\mathrm{st}}) \to (X, J)$ is given, where J is again continuous. Suppose further that $u(I_+) \subset W_+$ and $u(I_-) \subset W_-$, where W_+, W_- are totally real submanifolds of dimension $n = \frac{1}{2}\mathsf{dim}_{\mathbb{R}} X$ and intersect transversally. Note also, that transversality is understand here in a more general sense, see § 5 for details.

Theorem 4. *There is an* $\varepsilon^b > 0$ *such that if for all half-annuli* $R_k^+ := \{z \in \Delta^+ : e^{-(k+1)} \leqslant |z| \leqslant e^{-k}\}$ *one has* $\mathsf{area}\,(u(R_k^+)) \leqslant \varepsilon^b$*, then* u *extends to origin* $0 \in \Delta^+$ *as an* $L^{1,p}$*-map for some* $p > 2$*.*

As in the "inner" case, the necessary condition is weaker than the finiteness of energy. But unlike to "inner" and smooth boundary cases, it is possible that the map u in the last statement is $L^{1,p}$-regular in the neighborhood of "corner point" $0 \in \Delta^+$ only for some $p > 2$. For example, the map $u(z) = z^\alpha$ with $0 < \alpha < 1$ satisfies totally real boundary conditions $u(I_+) \subset \mathbb{R}$, $u(I_-) \subset e^{\alpha\pi i}\mathbb{R}$ and is $L^{1,p}$-regular only for $p < p^* := \frac{2}{1-\alpha}$. Note also, that by Sobolev imbedding $L^{1,p} \subset C^{0,\alpha}$ with $\alpha = 1 - \frac{2}{p}$, and thus u extends to zero at least continuously. In particular, $u(0) \in W_+ \cap W_-$.

One can see such a point x as a *corner point* for a corresponding pseudoholomorphic curve. Typical example appears in symplectic geometry when one takes Lagrangian submanifolds as boundary conditions.

The compactness theorem for open stable curves, stated in *Theorem 1*, was essentially used in [I-S1] and [I-S2] to describe envelopes of meromorphy of 2-spheres in algebraic surfaces.

The organization of the paper is the following. In §§ 1 and 2 we present, for the convenience of the reader, the basis notions concerning the topology on the space of stable curves and complex structure on the Teichmüller space of Riemann surfaces with boundary. In § 3 we give the necessary apriori estimates for the inner case, and in § 4 the prove of *Theorem 1*, related to curves with free boundary. This includes the case of closed curves. In § 5 we consider curves with totally real boundary conditions, obtain necessary apriori estimates at "totally real boundary", and prove the compactness theorem for such curves. In particular, we prove *Theorem 3* there.

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1. Stable curves and the Gromov topology

Before stating the Gromov compactness theorem, we need to introduce an appropriate category of pseudoholomorphic curves. Since in the limit of a sequence smooth curves one can obtain a singular one, a cusp-curve in the Gromov's terminology, we need to allow certain types of singularities of curves. On the other hand, it is desirable to have singularities as simple as possible.

A similar problem appears in looking for a "good" compactification of moduli spaces $\mathcal{M}_{g,m}$ of abstract complex smooth closed curves of genus g with m marked points. The Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,m}$, obtained by adding the *stable curves*, gives a satisfactory solution of this problem and suggests a possible way of generalization to other situations. In fact, the only singularity type one should allow are nodes, or nodal points. An appropriate notion for curves in a complex algebraic manifold X was introduced by Kontsevich in [K]. Our definition of stable curves over (X, J) is simply a translation of this notion to almost complex manifolds. The changee of terminology from *stable maps over* (X, J) to *stable curves* is motivated by the fact that we want to consider our objects as curves rather than maps.

Recall that a *standard node* is the complex analytic set $\mathcal{A}_0 := \{(z_1, z_2) \in \Delta^2 : z_1 \cdot z_2 = 0\}$. A point on a complex curve is called a *nodal point*, if it has a neighborhood biholomorphic to the standard node.

Definition 1.1. *A nodal curve C is a complex analytic space of pure dimension 1 with only nodal points as singularities.*

In other terminology, nodal curves are called *prestable*. We shall always suppose that C is connected and has a "finite topology", *i.e.* C has finitely many irreducible components, finitely many nodal points, and that C has a smooth boundary ∂C consisting of finitely many smooth circles γ_i, such that $\overline{C} := C \cup \partial C$ is compact.

Definition 1.2. *We say that a real oriented surface with boundary $(\Sigma, \partial\Sigma)$ parameterizes a complex nodal curve C if there is a continuous map $\sigma : \overline{\Sigma} \to \overline{C}$ such that:*

i) if $a \in C$ is a nodal point, then $\gamma_a = \sigma^{-1}(a)$ is a smooth imbedded circle in $\Sigma \backslash \partial\Sigma$, and if $a \neq b$ then $\gamma_a \cap \gamma_b = \varnothing$;

ii) $\sigma : \overline{\Sigma} \backslash \bigcup_{i=1}^{N} \gamma_{a_i} \to \overline{C} \backslash \{a_1, \dots, a_N\}$ is a diffeomorphism, where $a_1, \dots, a_N$ are the nodes of C.

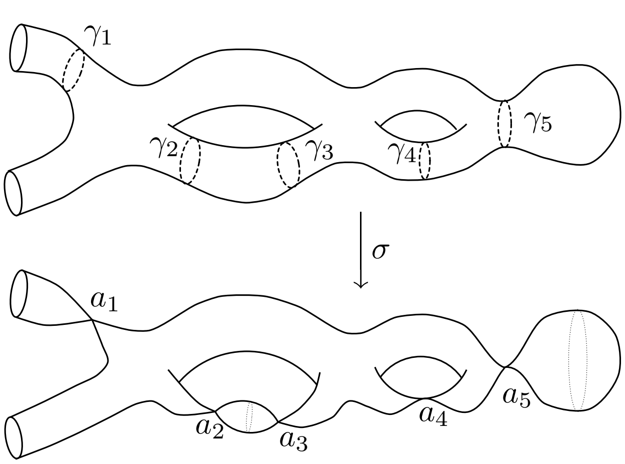

Fig. 1

Circles $\gamma_1, ..., \gamma_5$ are contracted by the parametrization map σ to nodal points $a_1, \dots a_5$.

Note that such a parametrization is not unique: if $g:\overline{\Sigma}\to\overline{\Sigma}$ is any orientation preserving diffeomorphism then $\sigma\circ g:\overline{\Sigma}\to\overline{C}$ is again a parametrization.

A parametrization of a nodal curve C by a real surface can be considered as a method of "smoothing" of C. An alternative method of "smoothing" — the normalization — is also useful for our purposes.

Consider the normalization $\hat{C}$ of C. Mark on each component of this normalization the pre-images (under the normalization map $\pi_C:\hat{C}\to C$) of nodal points of C. Let $\hat{C}_i$ be a component of $\hat{C}$. We can also obtain $\hat{C}_i$ by taking an appropriate irreducible component C_i, replacing nodes contained in C_i by pairs of disks with marked points, and marking remaining nodal points. Since it is convenient to consider components in this form, we make the following

Definition 1.3. *A component C' of a nodal curve C is a normalization of an irreducible component of C with marked points selected as above.*

This definition allows to introduce the Sobolev and Hölder spaces of functions and (continuous) maps of nodal curves. For example, a continuous map $u:C\to X$ is Sobolev $L^{1,p}_{\mathrm{loc}}$-smooth if so are all its restrictions on components of C. The most interesting case is, of course, the one of continuous $L^{1,2}_{\mathrm{loc}}$-smooth maps. In this case the energy functional $\|du\|^2_{L^2(C)}$ is defined. The definition of the energy $\|du\|^2_{L^2(C)}$ involves Riemannian metrics on X and C, which are supposed to be fixed.

Let C be a nodal curve and (X,J) an almost complex manifold with continuous almost complex structure J.

Definition 1.4. *A continuous map $u:C\to X$ is J-holomorphic if $u\in L^{1,2}_{\mathrm{loc}}(C,X)$ and*

$$du_x + J\circ du_x\circ j_C = 0 \tag{1.1}$$

for almost all $x\in C$. Here j_C denotes the complex structure on C.

The area of J-holomorphic map is defined as

$$\mathsf{area}\,(u(C)) := \|du\|^2_{L^2(C)}.$$

We shall show later that every J-holomorphic u is, in fact, $L^{1,p}_{\mathrm{loc}}(C,X)$-smooth for all $p<\infty$, see discussion after *Lemma 3.1*.

Remark. Our definition of the area uses the following fact. Let g be a Riemannian metric on C compatible with j_C, h a Riemannian metric on X, and $u:C\to X$ a J-holomorphic immersion. Then $\|du\|^2_{L^2(C)}$ is independent of the choice of g and coincides with the area of the image $u(C)$ w.r.t. the metric $h_J(\cdot,\cdot):=\frac{1}{2}(h(\cdot,\cdot)+h(J\cdot,J\cdot))$. The metric h_J here can be seen as a "Hermitization" of h w.r.t. J. The independence of $\|du\|^2_{L^2(C)}$ of the choice of a metric g on C in the same conformal class is a well-known fact, see *e.g.* [S-U]. Thus we can use the flat metric dx^2+dy^2 to compare the area and the energy. For J-holomorphic map we get

$$\|du\|^2_{L^2(C)} = \int_C |\partial_x u|^2_h + |\partial_y u|^2_h = \int_C |\partial_x u|^2_h + |J\partial_x u|^2_h = \int_C |du|^2_{h_J} = \mathsf{area}_{\,h_J}(u(C)),$$

where the last equality is another well-known result, see *e.g.* [G]. Since we consider changing almost complex structures on X, it is useful to know that we can use any Riemannian metric on X having reasonable notion of area.

Definition 1.5. *A stable curve over (X,J) is a pair (C,u), where C is a nodal curve and $u : C \to X$ is a J-holomorphic map, satisfying the following condition: If C' is a compact component of C, such that u is constant on C', then there exist finitely many biholomorphisms of C' which preserve the marked points of C.*

Remark. One can see that stability condition is nontrivial only in the following cases:

1) some component C' is biholomorphic to $\mathbb{CP}^1$ with 1 or 2 marked points; in this case u should be non-constant on any such component C';

2) some irreducible component C' is $\mathbb{CP}^1$ or a torus without marked points.

Since we consider only connected nodal curves, case *2)* can happen only if C irreducible, so that $C'=C$. In this case u must be non-constant on C.

Now we are going to describe the Gromov topology on the space of stable curves over X introduced in [G]. Let a sequence J_n of continuous almost complex structures on X be given, and suppose that J_n converge to J_∞ in the C^0-topology. Furthermore, let (C_n,u_n) be a sequence of stable curves over (X,J_n), such that all C_n are parametrized by the same real surface Σ.

Definition 1.6. *We say that (C_n,u_n) converges to a stable J_∞-holomorphic curve (C_∞,u_∞) over X if the parametrizations $\sigma_n : \overline{\Sigma} \to \overline{C}_n$ and $\sigma_\infty : \overline{\Sigma} \to \overline{C}_\infty$ can be chosen in such a way that the following holds:*

i) $u_n \circ \sigma_n$ converges to $u_\infty \circ \sigma_\infty$ in the $C^0(\Sigma,X)$-topology;

ii) if $\{a_k\}$ is the set of nodes of C_∞ and $\{\gamma_k\}$ are the corresponding circles in Σ, then on any compact subset $K \Subset \Sigma \backslash \cup_k \gamma_k$ the convergence $u_n \circ \sigma_n \to u_\infty \circ \sigma_\infty$ is $L^{1,p}(K,X)$ for all $p<\infty$;

iii) for any compact subset $K \Subset \overline{\Sigma} \backslash \cup_k \gamma_k$ there exists $n_0 = n_0(K)$ such that $\sigma_n^{-1}(\{a_k\}) \cap K = \varnothing$ for all $n \geqslant n_0$ and the complex structures $\sigma_n^* j_{C_n}$ converge smoothly to $\sigma_0^* j_{C_0}$ on K;

iv) the structures $\sigma_n^* j_{C_n}$ are constant in n near the boundary $\partial\Sigma$.

The reason for introducing the notion of a curve stable over X is similar to the one for the Gromov topology. We are looking for a completion of the space of smooth imbedded pseudoholomorphic curves which has "nice" properties, namely: 1) such a completion should contain the limit of a subsequence of every sequence of smooth curves, bounded in an appropriate sense; 2) such a limit should exist also for a subsequence of every sequence in the completed space; 3) such a limit should be unique. The Gromov's compactness theorem insures us that the space of curves stable over X enjoys these nice properties.

Condition *iv)* is trivial if Σ is closed, but it is important when one considers the "free boundary case", *i.e.* when Σ (and thus all C_n) are not closed and no boundary condition is imposed. However, we would like to point out that in our approach the "free boundary case" is essentially involved in the proof of compactness theorem also in the case of closed curves. On the other hand, in the case of curves with boundary on totally real submanifolds (see § 5) such a condition is unnecessary.

Recall that a complex annulus A has a conformal radius $R > 1$ if A is biholomorphic to $A(1,R) := \{z \in \mathbb{C} : 1 < |z| < R\}$.

Definition 1.7. *Let C_n be a sequence of nodal curves, parametrized by the same real surface Σ. We say that the complex structures on C_n do not degenerate near boundary, if there exist $R > 1$, such that for any n and any boundary circle $\gamma_{n,i}$ of C_n there exist an annulus $A_{n,i} \subset C_n$ adjacent to $\gamma_{n,i}$, such that all $A_{n,i}$ are mutually disjoint, do not contain nodal points of C_n, and have the same conformal radius R.*

Since conformal radius of all $A_{n,i}$ is the same, we can identify them with $A(1,R)$. This means that all changes of complex structures of C_n take place away from boundary. The condition is trivial if C_n and Σ are closed, $\partial\Sigma = \partial C_n = \varnothing$.

Remark. Changing our parametrizations $\sigma_n : \Sigma \to C_n$, we can suppose that for any i the pre-image $\sigma_n^{-1}(A_{n,i})$ is the same annulus A_i independent of n.

Now we state our main result. Fix some Riemannian metric h on X and some h-complete set $A \subset X$.

Theorem 1.1. *Let $\{(C_n, u_n)\}$ be a sequence of stable J_n-holomorphic curves over X with parametrizations $\delta_n : \Sigma \to C_n$. Suppose that:*

a) J_n are continuous almost complex structures on X, h-uniformly converging to J_∞ on A and $u_n(C_n) \subset A$ for all n;

b) there is a constant M such that $\mathsf{area}\,[u_n(C_n)] \leqslant M$ *for all n;*

c) complex structures on C_n do not degenerate near the boundary.

Then there is a subsequence (C_{n_k}, u_{n_k}) and parametrizations $\sigma_{n_k} : \Sigma \to C_{n_k}$, such that $(C_{n_k}, u_{n_k}, \sigma_{n_k})$ converges to a stable J_∞-holomorphic curve $(C_\infty, u_\infty, \sigma_\infty)$ over X.

Moreover, if the structures $\delta_n^* j_{C_n}$ are constant on the fixed annuli A_i, each adjacent to a boundary circle γ_i of Σ, then the new parametrizations σ_{n_k} can be taken equal to δ_{n_k} on some subannuli $A_i' \subset A_i$, also adjacent to γ_i.

Remarks. 1. In the proof, we shall give a precise description of convergence with estimates in neighborhoods of the contracted circles γ_i. The convergence of curves with boundary on totally real submanifolds will be studied in §5.

2. In applications, one uses a generalized version of the Gromov compactness theorem for nodal curves with marked point. This version is an immediate consequence of *Theorem 1.1* due to the following construction. Consider a nodal curve C and a J-holomorphic map $u : C \to X$. Let $\mathbf{x} := \{x_1, \ldots, x_m\}$ be the set of marked points on C which are supposed to be distinct from the nodal points of C. Define a new curve C^+ as the union of C with disks $\Delta_1, \ldots, \Delta_m$ such that $C \cap \Delta_i = \{x_i\}$ and such that any x_i becomes a nodal point of C^+. Extend f to a map $f^+ : C^+ \to X$ by setting $f^+|_{\Delta_i}$ to be constant and equal to $f(x_i)$. An appropriate definition of stability, used for triples $(C, \mathbf{x}, f)$, is equivalent to stability of (C^+, f^+). Similarly, the Gromov convergence $(C_n, \mathbf{x}_n, f_n) \to (C_\infty, \mathbf{x}_\infty, f_\infty)$ is equivalent to the Gromov convergence $(C_n^+, f_n^+) \to (C_\infty^+, f_\infty^+)$. Thus the Gromov compactness for curves with marked points reduces to the case considered in our paper. However, we shall consider curves with marked points as well.

In the rest of this section we shall describe topology and conformal geometry of nodal curves and compute the set of moduli parameterizing deformations of complex structure. As a basic reference we use the book of Abikoff [Ab].

Let C be a complex nodal curve parametrized by Σ.

Definition 1.8. *A component C' of C is called nonstable if one of the following two cases occurs:*

1) C' is $\mathbb{CP}^1$ and has one or two marked points;

2) C' is $\mathbb{CP}^1$ or a torus and has no marked points.

This notion of stability of abstract closed curves is due to Deligne-Mumford, see [D-M]. It was generalized by Kontsevich [K?] for the case of maps $f : C \to X$, *i.e.* for curves over X in our terminology. As it was already noted, the last case can happen only if $C = C'$. Strictly speaking, this case should be considered separately. However since such considerations require only obvious changes we just skip them and suppose that case *2)* does not occur.

Our first aim is to analyse the behavior of complex structures in the sequence (C_n, u_n) of J_n-holomorphic curves stable over X with uniformly bounded area, which are parametrized by the same real surface Σ. At the moment, uniform bound of area of $u_n(C_n)$ is needed only to show that the number of components of C_n is bounded. Passing to a subsequence, we can assume that all C_n are homeomorphic. This reduces the problem to a description of complex structures on a fixed nodal curve C.

To obtain such a description, it is useful to cut the curve into pieces where the behavior of complex structure is easy to understand. Such a procedure is a *partition into pants*. It is well known in the theory of moduli spaces of complex structure on curves, see *e.g.* [Ab], p. 93. Making use of it, we shall also do a slightly different procedure. Namely, we shall choose a special covering of Σ instead of its partition. Further, as blocks for our construction we shall use not only pants, but also disks and annuli. The reasons are that, firstly, the considered curves can have unstable components and, second, it is convenient to use annuli for a description of the deformation of the complex structure on curves. We start with

Definition 1.9. *An annulus A on a real surface or on a complex curve is a domain which is diffeomorphic (resp. biholomorphic) to the standard annulus $A(r, R) := \{z \in \mathbb{C} \;:\; r < |z| < R\}$, such that its boundary consists of smoothly imbedded circles. Pants (also called a pair of pants) on a real surface or on a complex curve is a domain which is diffeomorphic to a disk with 2 holes.*

The boundary of pants consists of three components, each of them being either a smoothly imbedded circle or a point. This point can be considered as a puncture of pants or as a marked point. An annulus or pants is *adjacent to a circle* γ if γ is one of its boundary components.

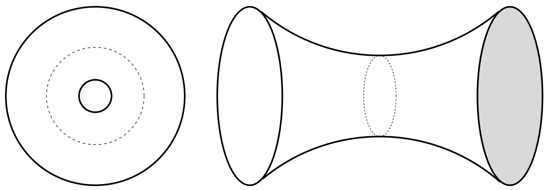

Fig. 2. An annulus

It is useful to imagine an annulus as a cylinder. After contracting the middle circle of the annulus we get a node.

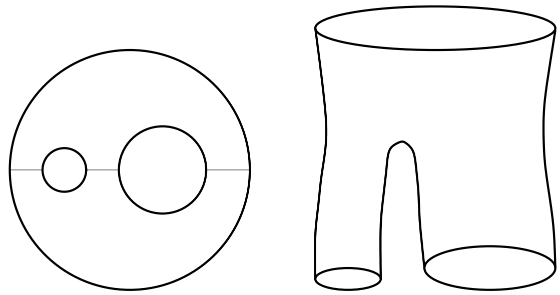

Fig. 3. Pants.

One can consider pants as a disc with two holes or as a sphere with three holes.

Let C be a nodal curve parametrized by a real surface Σ. We shall associate with every such curve C a certain graph Γ_C, which determines C topologically in a unique way. In fact, Γ_C will also determine a decomposition of some components of C into pants.

By the definition, a compact component C' is stable if it contains only a finite number of automorphisms preserving marking points. In this case $C'\backslash\{\textit{marked points}\}$ possesses a unique so-called intrinsic metric.

Definition 1.10. The *intrinsic metric* for a smooth curve C with marked points $\{x_i\}$ and with boundary ∂C is a metric g on $C\backslash\{\textit{marked points}\}$ satisfying the following properties:

i) g induces the given complex structure j_C;
ii) the Gauss curvature of g is constantly -1;
iii) g is complete in a neighborhood of every marked point x_i;
iv) every boundary circle γ of C is geodesic w.r.t. g.

Note that such a metric, if exists, is unique, see *e.g.* [Ab].

Now consider a component C' of C adjacent to some boundary circle of C. Then $C'\backslash\{\textit{marked points}\}$ is either

a) a disk Δ, or
b) an annulus A, or
c) a punctured disk $\check{\Delta}$, or else
d) $C'\backslash\{\textit{marked points}\}$ admits the intrinsic metric.

Note that if a component C' is a disk or an annulus (both without marked points), then C' is the whole curve C. We shall considered cases a) and b) later. Now we assume for simplicity that cases a) and b) do not occur.

Definition 1.11. *A component C' of a nodal curve C is non-exceptional if $C'\backslash\{\textit{marked points}\}$ admits the intrinsic metric.*

In particular, nonstable components are exceptional compact ones, and exceptional non-compact components are those of following types a)–c) above.

Take some non-exceptional component C' of C. There is a so called maximal partition of $C'\backslash\{\textit{marked points}\}$ into pants $\{C_1, \dots C_n\}$, such that all boundary components of these pants are either simple geodesics circles in intrinsic metric or marked points, see [Ab]. Let us fix such a partition and mark the obtained geodesics circles on C'.

Let now $\sigma : \Sigma \to C$ be some parametrization of C. This defines the set $\boldsymbol{\gamma}'$ of the circles on Σ which correspond to the nodes of C. Let $\boldsymbol{\gamma}''$ be the set of σ-preimages of the geodesics, chosen above. Then $\boldsymbol{\gamma} := \boldsymbol{\gamma}' \sqcup \boldsymbol{\gamma}''$ forms a system of disjoint "marked" circles on Σ, which encodes the topological structure of C. Now the graph Γ_C in question can be constructed as follows.

Define the set V_C of vertices of Γ_C to be the set $\{S_j\}$ of connected components of $\Sigma \backslash \cup_{\gamma \in \boldsymbol{\gamma}} \gamma = \sqcup_j S_j$. Any $\gamma \in \boldsymbol{\gamma}$ lies between 2 components, say S_j and S_k, and we draw an edge connecting the corresponding 2 vertices. Further, any boundary circle γ of Σ has the uniquely defined component S_j adjacent to γ. For any such γ we draw a *tail*, *i.e.* an edge with one end free, attached to vertex S_j. Finally, we mark all edges which correspond to the circles $\boldsymbol{\gamma}'$, *i.e.* those coming from from nodes.

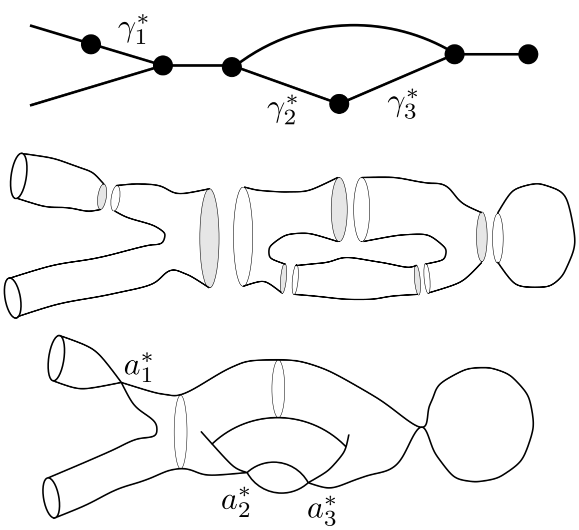

Fig. 4. Graph of a curve C.

Graph Γ_C determines the topology of the curve C in a unique way. Take as many oriented spheres as many vertices Γ_C has. For each edge take a handle and join the corresponding spheres by this handle. For each tail make a hole (*i.e.* remove a disk) in the corresponding sphere. Finally, contract into points the circles on the handles corresponding to the marked edges to get nodes. We obtain a topological space homeomorphic to C.

Having the graph Γ, which characterizes uniquely the topological structure of C, we are now going to describe the set of parameters, defining (uniquely) the complex structure of the curves C. This is equivalent to determining the complex structure and marked points on all components of C. If such a component C' is a sphere with 1 or 2 marked points or a disk with 1 marked point, then its structure is defined by its topology uniquely up to diffeomorphism. Otherwise, the component C' is non-exceptional. In this case the complex structure and the marked points can be restored by the so called *Fenchel-Nielsen coordinates* on the Teichmüller space $\mathbb{T}_{g,m,b}$. Recall that the space $\mathbb{T}_{g,m,b}$ parametrizes the complex structures on a Riemann surface Σ of genus g with m punctures (*i.e.* marked points) and with boundary consisting of b circles, see [Ab].

Let C be a smooth complex curve with marked points of non-exceptional type, so that C admits the intrinsic metric. Fix some parametrization $\sigma : \Sigma \to C$. Consider the preimages of the marked points on C as marked points on Σ or, equivalently, as punctures of Σ. Let $C \backslash \{\textit{marked points}\} = \cup_j C_j$ be a decomposition of C into pants and $\Sigma \backslash \{\textit{marked points}\} = \cup_j S_j$ the induced decomposition of Σ.

Let $\{\gamma_i\}$ be the set of boundary circles of Σ. The boundary of every pants S_j has three components, each of them being either a marked point of Σ or a circle. In the

last case this circle is either a boundary component of Σ or a boundary component of another pants, say S_k. In this situation we denote by γ_{jk} the circle lying between the pants S_j and S_k. Fix the orientation on γ_{jk}, induced from S_j if $j<k$ and from S_k if $k<j$. For any such circle γ_{jk}, fix a boundary component of S_j different from γ_{jk} and denote it by $\partial_k S_j$. In the same way fix a boundary component $\partial_j S_k$. Make similar notations on C using primes to distinguish the circles on C from those on Σ, *i.e.* set $\gamma'_i := \sigma(\gamma_i)$ and $\gamma'_{jk} := \sigma(\gamma_{jk})$.

By our construction, $\gamma'_{jk} = \sigma(\gamma_{jk})$ is a geodesic w.r.t. intrinsic metric in C.

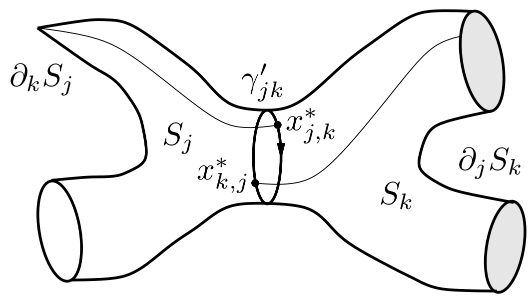

Fig. 5. Marked points on the circle γ'_{jk}.

If the component $\partial_k C_j$ is a marked point, we find on C_j the (uniquely defined) geodesic ray $\alpha_{j,k}$ starting at some point $x^*_{j,k} \in \gamma'_{jk}$ and approaching $\partial_k C_j$ at infinity, such that $\alpha_{j,k}$ has no self-intersections and is orthogonal to γ'_{jk} at $x^*_{j,k}$. Otherwise, we find on C_j the shortest geodesic $\alpha_{j,k}$ which connects $\partial_k C_j$ with γ'_{jk} and denote the point $\alpha_{j,k} \cap \gamma'_{jk}$ by $x^*_{j,k}$. In both cases, this construction detemies a distinguished point $x^*_{j,k} \in \gamma'_{jk}$.

Doing the same procedure in C_k, we obtain another point $x^*_{k,j} \in \gamma'_{jk}$. Denote by ℓ_{jk} (resp. by ℓ_i) the intrinsic length of γ'_{jk} (resp. of $\gamma'_i := \sigma(\gamma_i)$) in C. For $j<k$ define λ_{jk} as the intrinsic length of the arc on γ'_{jk}, which starts at $x^*_{j,k}$ and goes to $x^*_{k,j}$ in the direction determined by the orientation of γ_{jk}. Set $\vartheta_{jk} := \frac{2\pi\lambda_{jk}}{\ell_{jk}}$. We shall consider ϑ_{jk} as a function of the complex structure j_C on C with values in $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$.

The parameters $\boldsymbol{\ell} := (\ell_i, \ell_{jk})$ and $\boldsymbol{\vartheta} := (\vartheta_{jk})$ are called *Fenchel-Nielsen coordinates* of the complex structure j_C. The reason is that these parameters determines up to isomorphism the complex structure j_C on the smooth complex curve with marked points parametrized by a real surface Σ. In other words, $(\boldsymbol{\ell}, \boldsymbol{\vartheta})$ can be considered as coordinates on $\mathbb{T}_{g,m,b}$. More precisely, one has the following

Proposition 1.2. *Let Σ be a real surface of genus g with m marked points and with the boundary consisting of b circles, so that $2g+m+b \geqslant 3$. Let $\Sigma \backslash \{$marked points$\} = \cup_j S_j$ be its decomposition into pants. Then*

i) for any given tuples $\boldsymbol{\ell} = (\ell_i, \ell_{jk})$ and $\boldsymbol{\vartheta} = (\vartheta_{jk})$ with $\ell_i, \ell_{jk} > 0$ and $\vartheta_{jk} \in S^1$ there exists a complex structure j_C on Σ, such that boundary circles of all S_j are geodesic w.r.t. the intrinsic metric on $\Sigma \backslash \{$marked points$\}$ defined by j_C, and such that the given $(\boldsymbol{\ell}, \boldsymbol{\vartheta})$ are Fenchel-Nielsen coordinates of j; moreover, such a structure j_C is unique up to a diffeomorphism preserving the pants S_j and the marked points;

ii) let C be a smooth complex curve with parametrization $\sigma : \Sigma \to C$ which has m marked points; then there exists a parametrization $\sigma_1 : \Sigma \to C$ isotopic to σ, which maps boundary components and marked points of Σ onto the ones of C in prescribed order, and such that the boundary circles of $\sigma(S_j)$ are geodesic w.r.t. the intrinsic metric on $C \backslash \{$marked points$\}$.

Proof. See [Ab]. $\square$

2. Complex structure on the space $\mathbb{T}_\Gamma$

Let Σ be a real oriented surface of genus g with m marked points and with the boundary consisting of b circles. Assume that $2g+m+b \geqslant 3$. Then there exists a decomposition of $\Sigma\backslash\{\textit{marked points}\}$ into pants, which is in general not unique. The topological type of such a decomposition can be encoded in graph Γ, associated with the decomposition. It is constructed in similar way as above, but this time we must draw a tail also for every marked point, and then mark all those tails on the graph.

Let such a graph Γ be fixed. We call two complex structures J_1 and J_2 on Σ isomorphic if there exists a biholomorphism $\varphi : (\Sigma, J_1) \cong (\Sigma, J_1)$ preserving the marked points of Σ and the decomposition of Σ into pants given by graph Γ. Denote by $\mathbb{T}'_\Gamma$ the space of isomorphism classes of complex structures on Σ. By *Proposition 1.2*, Fenchel-Nielsen coordinates identify $\mathbb{T}'_\Gamma$ with the real manifold $\mathbb{R}_+^{3g-3+m+2b} \times (S^1)^{3g-3+m+b}$.

It is desirable to equip $\mathbb{T}'_\Gamma$ with some natural complex structure. In doing so, the main difficulty is that the real dimension of $\mathbb{T}'_\Gamma$ can be odd. A possible explanation of this fact is that not all relevant information (*i.e.* parameters) about a complex structure has been taken into consideration. Note that for any "inner circle" γ_{jk} which appears after the decomposition into pants we have obtained a pair of coordinates, manely the length ℓ_{jk} and the angle ϑ_{jk}. On the other hand, for any boundary circle γ_i of Σ we have got only the length ℓ_i. An obvious way to produce additional angle coordinates is to introduce an additional marking of every boundary circle.

Definition 2.1. *A real surface Σ or a nodal complex curve C is said to have a marked boundary if on every boundary circle of Σ (resp. C) a point is fixed.*

Remark. Later in § 5 we shall consider complex curves with several marked points on boundary circles. But now we shall assume that on every boundary circle exactly one point is marked.

"Missed" angle coordinates ϑ_i can be now introduced similarly to ϑ_{jk}. For a boundary circle γ_i we consider the adjacent pants S_j. Fix a boundary component $\partial_i S_j$ different from γ_i. Let J be a complex structure on Σ such that boundary circles of all pants S_k are geodesic w.r.t. the intrinsic metric defined by J. Using constructions from above, find a geodesic (resp. a ray) α_i starting at point $x_i^* \in \gamma_i$ and ending at boundary circle $\partial_i S_j$ (resp. approaching marked point $\partial_i S_j$ of Σ). Take the marked boundary point ζ_i on γ_i and consider the length λ_i of the geodesic ark on γ_i, starting at x_i^* and going to ζ_i in the direction defined by the orientation of γ_i. Define $\vartheta_i := \frac{2\pi\lambda_i}{\ell_i}$, $\vartheta_i \in S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$. We include the coordinates ϑ_i into the system of angle coordinates $\boldsymbol{\vartheta}$. Denote by $\mathbb{T}_\Gamma$ the set of isomorphism classes of complex structures on Σ with marked boundary and with a given decomposition into pants.

Let C be a smooth complex curve with marked points and a marked boundary, $C\backslash\{\textit{marked points}\} = \cup_j C_j$ its decomposition into pants, $\sigma : \Sigma \to C$ a parametrization, and $\Sigma\backslash\{\textit{marked points}\} = \cup_j S_j$ the induced decomposition of Σ. To define a complex structure on $\mathbb{T}_\Gamma$, we introduce special local holomorphic coordinates in a neighborhood of boundary of pants on C. Consider some pants S_j and its boundary

circle γ^*. It can be a boundary circle of Σ, γ_i in our previous notation, or a circle γ_{jk} separating S_j from another pants S_k. Let ℓ^* be the intrinsic length of γ^*. Fix some small $a>0$ and consider the annulus A consisting of those $x \in S_j$ for which the intrinsic distance $\mathsf{dist}(x,\gamma^*) < a$. The universal cover $\tilde A$ can be imbedded into hyperbolic plane $\mathbb{H}$ as an infinite strip Θ of constant width a, such that one of its borders is geodesic line L. The action of a generator of $\pi_1(A) \cong \mathbb{Z}$ on $\tilde A$ is defined by the shift of Θ along L by distance ℓ^*.

Now consider the annulus $A' := [0, \frac{\pi^2}{\ell^*}[\times S^1$ with coordinates ρ, θ, $0 \leqslant \rho < \frac{\pi^2}{\ell^*}$, $0 \leqslant \theta \leqslant 2\pi$ and with the metric $(\frac{\ell^*}{2\pi}/\mathsf{cos}\frac{\ell^*\rho}{2\pi})^2(d\rho^2 + d\theta^2)$. A direct computation shows that this metric is of constant curvature -1 and that boundary circle $\partial_0 A_1 := S^1 \times \{0\}$ is geodesic of length ℓ^*, whereas A' is complete in a neighborhood of the other boundary circle. Consequently, the universal cover $\tilde A'$ of A' can be imbedded in the hyperbolic plane $\mathbb{H}$ as a hyperbolic half-plane $\mathbb{H}^+_L$ with a boundary line L, such that $\Theta \subset \mathbb{H}^+_L$. Moreover, the action of $\pi_1(A') \cong \mathbb{Z}$ on $\tilde A' \cong \mathbb{H}^+_L$ is the same as for $\tilde A \cong \Theta$. This shows that there exists an *isometric* imbedding of A into A' which maps γ^* onto $\partial_0 A'$. Moreover, such an imbedding is unique up to rotations in the coordinate θ. This leads us to the following

Proposition 2.1. *Let C_j be pants with a complex structure and γ^* its boundary circle of the intrinsic length ℓ^*. Let x^* be a point on γ. Then some collar annulus A of γ^* possesses the uniquely defined conformal coordinates $\theta \in S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$ and ρ, such that the intrinsic metric has the form $(\frac{\ell^*}{2\pi}/\mathsf{cos}\frac{\ell^*\rho}{2\pi})^2(d\rho^2 + d\theta^2)$, $\rho|_{\gamma^*} \equiv 0$, $\theta(x^*) = 0$, and such that the orientation on S_j is given by $d\theta \wedge d\rho$.*

We shall represent ρ and θ also in the complex form $\zeta := e^{-\rho + i\theta}$ and call ζ the *intrinsic coordinate* of the pants C_j at γ^*. An important corollary of the description of the intrinsic metric in a neighborhood of boundary circle is the following statement about non-degenerating complex structures in pants, see *Definition 1.7.*

Lemma 2.2. *Let C be a smooth complex curve with marked points admitting the intrinsic metric and let γ^* be a boundary circle of C of length ℓ^*.*

i) If there exists an annulus $A \subset C$ of conformal radius R (i.e. $A \cong \{z \in \mathbb{C} : 1 < |z| < R\}$), adjacent to γ^* and containing no marked points, then $\mathsf{log} R \leqslant \frac{\pi^2}{\ell^*}$

ii) There exists a universal constant a^* such that the condition $\ell^* \leqslant 1$ implies that there exist an annulus $A \subset C$ of conformal radius R with $\mathsf{log} R \geqslant \frac{\pi^2}{\ell^*} - \frac{2\pi}{a^*}$, which is adjacent to γ^*, has area a^* and contains no marked points of C.

iii) Let $\gamma \subset C$ be a simple geodesic circle of the length ℓ and $A \subset C \backslash \{marked\ points\}$ annulus of conformal radius R homotopy equivalent to γ. Then $\mathsf{log} R \leqslant \frac{2\pi^2}{\ell}$.

Proof. Let Ω be the universal cover of $C \backslash \{marked\ points\}$ equipped with the intrinsic metric lifted from C. Then Ω can be isometrically imbedded into the hyperbolic plane $\mathbb{H}$ as a domain bounded by geodesic lines, such that each of these lines covers some boundary circle γ_i. Take some (not unique!) line L covering the circle γ^* and fix a hyperbolic half-plane $\mathbb{H}^+_L$ with a boundary line L, so that $\Omega \subset \mathbb{H}^+_L$.

Now consider the universal cover $\tilde A$ of the annulus A and provide it with the metric induced from C. Then we can isometrically imbed $\tilde A$ in $\mathbb{H}^+_L$ in such a way that the line covering $\gamma^* \subset \partial A$ will be mapped onto L. The action of a generator of $\pi_1(A) \cong \mathbb{Z}$ on $\tilde A$ is defined by the shift of $\mathbb{H}$ along L onto distance

ℓ^*. Consequently, A can be isometrically imbedded into $\mathbb{H}^+_L/\pi_1(A)$, which is the annulus $A' = [0, \frac{\pi^2}{\ell^*}[\times S^1$ with coordinates ρ, θ, $0 \leqslant \rho < \frac{\pi^2}{\ell^*}$, $0 \leqslant \theta \leqslant 2\pi$ and with metric $(\frac{\ell^*}{2\pi}/\mathsf{cos}\frac{\ell^*\rho}{2\pi})^2(d\rho^2 + d\theta^2)$. Note that the conformal radius of A' is e^{π^2/ℓ^*}. The monotonicity of the conformal radius of annuli (see *e.g.* [Ab], Ch.II, §1.3) yields the inequality $R \leqslant e^{\pi^2/\ell^*}$ which is equivalent to first assertion of the lemma.

Part *iii)* of the lemma can be proved by same argument. More precisely, under the hypothesis of part *iii)* we imbed the annulus A into the annulus $A'' =]-\frac{\pi^2}{\ell}, \frac{\pi^2}{\ell}[\times S^1$ with coordinates ρ, θ, $-\frac{\pi^2}{\ell} < \rho < \frac{\pi^2}{\ell}$, $0 \leqslant \theta \leqslant 2\pi$ and with metric $(\frac{\ell}{2\pi}/\mathsf{cos}\frac{\ell\rho}{2\pi})^2(d\rho^2 + d\theta^2)$. The conformal radius of A is now estimated by the conformal radius of A'', which is equal to $e^{\frac{2\pi^2}{\ell}}$.

The second part of our lemma follows from results of Ch.II, § 3.3 of [Ab]. *Lemma 2* there says that there exists a universal constant a^* with the following property: If $\ell^* \leqslant 1$, then there exists a collar neighborhood A of constant width ρ^* and of area a^*, which is an annulus imbedded in C and contains no marked points of C. In particular, we can extend the intrinsic coordinates ρ and θ in A. Using these coordinates, we present A in the form $\{(\rho, \theta) : 0 \leqslant \rho \leqslant \rho^*\}$ and compute the area,

$$a^* = \mathsf{area}\, A = 2\pi \int_{\rho=0}^{\rho^*} \left(\frac{\ell^*/2\pi}{\mathsf{cos}(\ell^*\rho/2\pi)} \right)^2 d\rho = \ell^* \mathsf{tan} \left(\frac{\ell^*\rho^*}{2\pi} \right).$$

Consequently, $\mathsf{tan}\left(\frac{\pi}{2} - \frac{\ell^*\rho^*}{2\pi}\right) = \mathsf{cotan}\left(\frac{\ell^*\rho^*}{2\pi}\right) = \frac{\ell^*}{a^*}$. This implies $\frac{\pi}{2} - \frac{\ell^*\rho^*}{2\pi} \leqslant \frac{\ell^*}{a^*}$, which is equivalent to $\rho^* \geqslant \frac{\pi^2}{\ell^*} - \frac{2\pi}{a^*}$. To finish the proof we note that the conformal radius R of A is equal e^{ρ^*}. □

Let C be a smooth complex curve with marked points, C_j a piece of a decomposition of $C \backslash \{$*marked points*$\}$ into pants and γ^* its boundary circle. Then as a "base point" $x^* = \{\theta = 0 = \rho\}$ for the definition of the intrinsic coordinate we shall use the point $x^*_{j,k}$ if $\sigma(\gamma^*)$ is the geodesic separating C_j from another pants C_k, or respectively the point x^*_i if γ^* is a boundary circle of C. We denote these coordinates $\zeta_{j,k} = e^{-\rho_{j,k} + i\theta_{j,k}}$ and $\zeta_i = e^{-\rho_i + i\theta_i}$. Note that ϑ_i is exactly the θ-coordinate of the marked boundary point $x_i \in \gamma_i$ with respect to x^*_i and ϑ_{jk} is the θ-coordinate of $x^*_{k,j}$ with respect to $x^*_{j,k}$.

Note also that any intrinsic coordinate of a pair $(\zeta_{j,k}, \zeta_{k,j})$ extends canonically from one collar neighborhood of γ_{jk} to another side in such a way that the formula for the intrinsic metric remains valid. This extension possesses the property $\zeta_{j,k} \cdot \zeta_{k,j} \equiv e^{i\vartheta_{jk}}$ where ϑ_{jk} a constant function. We can view this relation as the transition function from $\zeta_{j,k}$ to $\zeta_{k,j}$.

A similar construction is possible in the case of a boundary circle γ_i. Namely, allowing ρ_i to change also in the interval $]-\frac{\pi^2}{\ell_i}, 0]$ and maintaining the formula $(\frac{\ell_i}{2\pi}/\mathsf{cos}\frac{\ell_i\rho_i}{2\pi})^2(d\rho_i^2 + d\theta_i^2)$ for the metric we can glue to Σ an annulus $]-\frac{\pi^2}{\ell_i}, 0] \times S^1$ and extend there the coordinate $\zeta_i = e^{-\rho_i + i\theta_i}$.

Making such a construction with every boundary circle γ_i we obtain a complex curve $C^{(N)}$ with the following properties. C is relatively compact in $C^{(N)}$ and the intrinsic metric of C extends to a complete Riemannian metric on $C^{(N)}$ with constant curvature -1. Such the extension and the metric are unique. $C^{(N)}$ is called

the *Nielsen extension of* C, see [Ab]. Note that the complex coordinate ζ_i can be extended further to the unit disk $\{|\zeta_i|<1\}$.

Using the introduced complex coordinates ζ_i and $\zeta_{j,k}$, we define a deformation family of complex structures on the curve C with marked boundary. Let λ_i and λ_{jk} be complex parameters changing in small neighborhoods of $e^{i\theta_i}$ and $e^{i\theta_{jk}}$ respectively. Having these data $\boldsymbol{\lambda}=(\lambda_i,\lambda_{jk})$, construct a complex curve $C_{\boldsymbol{\lambda}}$ in the following way. Take the pants $\{C_j\}$ of the given decomposition of C and extend all the complex coordinates ζ_i and ζ_{jk} outside the pants. Glue the pairs of coordinates $(\zeta_{j,k},\zeta_{k,j})$ with new transition relations $\zeta_{j,k}\cdot\zeta_{k,j}=\lambda_{jk}$ (constant functions). Move original boundary circles $\gamma_i=\{|\zeta_i|=1\}$ of C to new positions defined by the equations $|\zeta_i|=|\lambda_i|$ and mark the points $\zeta_i=\lambda_i$ on them.

Theorem 2.3. *The natural map* $F:\boldsymbol{\lambda}\to(\boldsymbol{\ell},\boldsymbol{\vartheta})$ *is non-degenerated. In particular,* $\boldsymbol{\lambda}$ *can be considered as the set of local complex coordinates on* $\mathbb{T}_\Gamma$ *and* $\mathcal{C}:=\{C_{\boldsymbol{\lambda}}\}$ *as a (local) universal holomorphic family of curves over* $\mathbb{T}_\Gamma$.

Proof. Write the functions $\boldsymbol{\lambda}=(\lambda_i,\lambda_{jk})$ in the form $\lambda_i=e^{-r_i+i\varphi_i}$, $\lambda_{jk}=e^{-r_{jk}+i\varphi_{jk}}$. From the definition of the map

$$F:(e^{-r_i+i\varphi_i},e^{-r_{jk}+i\varphi_{jk}})\mapsto(\ell_i,\ell_{jk};\vartheta_i,\vartheta_{jk})$$

it is easy to see that $\frac{\partial(\vartheta_i,\vartheta_{jk})}{\partial(\varphi_i,\varphi_{jk})}$ is the identity matrix, whereas $\frac{\partial(\ell_i,\ell_{jk})}{\partial(\varphi_i,\varphi_{jk})}$ is equal to 0. So it remains to show that the matrix $\frac{\partial(\ell_i,\ell_{jk})}{\partial(r_i,r_{jk})}$ is non-degenerate.

Consider a special case when C is pants with the boundary circles γ_i (at least one) and, possibly, with marked points x_j. We shall consider such points as punctures of C. Let J denote the complex structure on C and let μ_0 be the intrinsic metric. Extend the coordinates ζ_i and the metric μ_0 outside of γ_i to some bigger complex curve $\widetilde{C}$ with $C\Subset\widetilde{C}$.

Fix real numbers v_i and consider the domains C_t in $\widetilde{C}$ defined in local coordinates $\zeta_i=e^{-\rho_i+i\theta_i}$ by inequalities the $\rho_i\geqslant v_it$. This defines a family of deformations of $C=C_0$ parametrized by a real parameter t, corresponding to a real curve in the parameter space $\{\boldsymbol{\lambda}\}$ given by $\lambda_i(t)=e^{v_it}$. Note that the deformation is made in such a way that original complex structure J and local holomorphic coordinates are preserved. Thus we can use them as "invariable basis" in our calculations.

Let μ_t be the intrinsic metric of C_t. Without loss of generality we may assume that μ_t extends to $\widetilde{C}$ as a metric with constant curvature -1, which induces the original complex structure J on $\widetilde{C}$. Since the intrinsic metric depends smoothly on the operator J of complex structure, μ_t are smooth in t. In a local holomorphic coordinate $z=x+iy$ we can present μ_t in the form $e^{2\psi(t,z)}(dx^2+dy^2)$. The condition $\mathsf{Curv}(\mu_t)\equiv-1$ is equivalent to the differential equation

$$\partial^2_{xx}\psi(t,\cdot)+\partial^2_{yy}\psi(t,\cdot)=e^{2\psi(t,\cdot)}$$

where ∂_x denotes the partial derivation $\frac{\partial}{\partial x}$ and so on. Differentiating it in t we get $e^{-2\psi(t,\cdot)}(\partial^2_{xx}+\partial^2_{yy})\dot\psi(t,\cdot)=\dot\psi(t,\cdot)$, where $\dot\psi(t,\cdot)$ denotes the derivative of $\psi(t,\cdot)$ in t.

Note that $\partial_t\mu_t=2\dot\psi(t,\cdot)\cdot\mu_t$, so $\dot\psi_t(\cdot)=\dot\psi(t,\cdot)$ is independent of the choice of a local holomorphic coordinate $z=x+iy$ and is defined globally. The equation

$e^{-2\psi_z(t,\cdot)}(\partial^2_{xx}+\partial^2_{yy})\dot\psi_z(t,\cdot)=\dot\psi_z(t,\cdot)$ can be rewritten in the form $\Delta_t\dot\psi_t=2\dot\psi_t$, with Δ_t denoting the Laplace operator for the metric μ_t.

The condition that the circle $\gamma_i(t):=\{\rho_i=v_it\}$ is μ_t-geodesic means that the covariant derivative $\nabla_{\theta_i}(\partial_{\theta_i})$ of the vector field ∂_{θ_i}, the tangent vector field to $\gamma_i(t)$, must be parallel to ∂_{θ_i} along $\gamma_i(t)$. Expressing this relation in local coordinates ρ_i and θ_i, we get $\partial_{\rho_i}\psi_i(t;v_it,\theta_i)=0$, where $\mu_t=e^{2\psi_i(t;\rho_i,\theta_i)}(\partial\rho_i^2+\partial\theta_i^2)$ is a local representation of the metric μ_t. Deriving in t, we get $\partial_{\rho_i}\dot\psi_i(t;v_it,\theta_i)+v_i\partial^2_{\rho_i\rho_i}\psi_i(t;v_it,\theta_i)=0$.

In the case $t=0$ we have $\psi_i(0;0,\theta_i)\equiv\log\frac{\ell_i}{2\pi}$, a constant. Hence $\partial^2_{\theta_i\theta_i}\psi_i(0;0,\theta_i)=e^{2\psi_i(0;0,\theta_i)}$ and $\partial_{\rho_i}\dot\psi_i(0;0,\theta_i)=-v_ie^{2\psi_i(0;0,\theta_i)}=-v_i\left(\frac{\ell_i}{2\pi}\right)^2$. On the other hand, $\partial_{\rho_i}=-\frac{\ell_i}{2\pi}\partial_\nu$ on $\gamma_i(0)=\gamma_i$ where ν denotes the unit outer normal field to $C_0=C$. Consider the integral $\int_C|d\dot\psi_0|^2+2\dot\psi_0^2d\mu_0$. Integrating by parts, we get

$$
\begin{aligned}
&\int_C|d\dot\psi_0|^2+2\dot\psi_0^2d\mu_0=\int_C\dot\psi_0(2\dot\psi_0-\Delta_0\dot\psi_0)d\mu_0+\int_{\partial C}\dot\psi_0\partial_\nu\dot\psi_0dl=\\
&=\sum_i\int_{\gamma_i}\dot\psi_0\partial_\nu\dot\psi_0\frac{\ell_i}{2\pi}d\theta_i=\sum_i-\int_{\gamma_i}\dot\psi_0\partial_{\rho_i}\dot\psi_0d\theta_i=\sum_i\int_{\gamma_i}\dot\psi_0v_i\left(\frac{\ell_i}{2\pi}\right)^2d\theta_i=\\
&=\sum_i\frac{v_i\ell_i}{2\pi}\int_{\gamma_i}\dot\psi_0e^{\psi_i(0;0,\theta_i)}d\theta_i=\sum_i\frac{v_i\ell_i}{2\pi}\int_{\gamma_i}(\dot\psi_0+v_i\partial_{\rho_i}\psi_i(0;0,\theta_i))e^{\psi_i(0;0,\theta_i)}d\theta_i=\\
&=\sum_i\frac{v_i\ell_i}{2\pi}\frac{\partial}{\partial t}\Big|_{t=0}\int_{\gamma_i}e^{\psi_i(t;v_it,\theta_i)}d\theta_i=\sum_i\frac{v_i\ell_i}{2\pi}\frac{\partial}{\partial t}\Big|_{t=0}\ell_i(t)=\sum_i\frac{\ell_i}{2\pi}v_i\dot\ell_i.
\end{aligned}
$$

Here $\dot\ell_i$ denotes the derivative of the length parameter ℓ_i for the curve C_t at $t=0$, so that $(\dot\ell_1,\dot\ell_2,\dot\ell_3)=dF(v_1,v_2,v_3)$. The obtained relation shows that the Jacobi matrix $dF=\frac{\partial(\ell_i,\ell_2,\ell_3)}{\partial(r_1,r_2,r_3)}$ is non-degenerate. Otherwise there would exist a nonzero vector (v_1,v_2,v_3) such that for the deformation constructed above we get $\dot\ell_i=0$. But then $\dot\psi_0\equiv0$, which is a contradiction.

Now consider a general situation. Let Σ be a real surface with marked boundary, C a smooth curve with marked points, $\sigma:\Sigma\to C$ a parametrization, and $C\backslash\{\textit{marked points}\}=\cup_jC_j$ a decomposition into pants with a given graph Γ. Let $\{\gamma_i\}$ be the set of boundary circles and $\{\gamma_{jk}\}$ the set of circles lying between the pants C_j and C_k respectively. Consider these pants separately. Then for any circle $\gamma_{jk}=\gamma_{kj}$ we obtain 2 distinguished ones, $\gamma_{j,k}$ considered as a boundary circle of C_j, and $\gamma_{k,j}$ considered as a boundary circle of C_k. Take real numbers $\boldsymbol{v}:=(v_i,v_{j,k},v_{k,j})$ where v_i is associated with the circle γ_i, $v_{j,k}$ with $\gamma_{j,k}$, and $v_{k,j}$ with $\gamma_{k,j}$ respectively. Let $C_j(t\boldsymbol{v})$ denote the pants obtained from C_j by the above construction using the corresponding parameters v_i and $v_{j,k}$. For $\boldsymbol{v}$ lying in a small ball $B=\{|\boldsymbol{v}|<\varepsilon\}$ all such families $C_j(t\boldsymbol{v})$ can be extended for all $t\in[-1,1]$. Thus over B we obtain a collection of deformation families $C_j(\boldsymbol{v})$ of complex structure on pants C_j.

Let $\ell_i(\boldsymbol{v})$, $\ell_{j,k}(\boldsymbol{v})$, and $\ell_{k,j}(\boldsymbol{v})$ denote the lengths of circles γ_i, $\gamma_{j,k}$, and $\gamma_{k,j}$ w.r.t. obtained intrinsic metrics $\mu_j(\boldsymbol{v})$ on $C_j(\boldsymbol{v})$. Denote by $\dot\ell_i$ a linear functional $\partial_t|_{t=0}\ell_i(t\boldsymbol{v})$, and define $\dot\ell_{j,k}$ similarly. The explicit formula for an intrinsic metric near a boundary circle showss that $C_j(\boldsymbol{v})$ can be glued to $C_k(\boldsymbol{v})$ along γ_{jk} exactly

when $\ell_{j,k}(\boldsymbol{v}) = \ell_{k,j}(\boldsymbol{v})$. Since the Jacobian $\frac{\partial \boldsymbol{\ell}(\boldsymbol{v})}{\partial \boldsymbol{v}}$ is non-degenerate, the conditions $\ell_{j,k}(\boldsymbol{v}) = \ell_{k,j}(\boldsymbol{v})$ define a submanifold $V \subset B$ whose tangent space T_0V is given by relations $\dot{\ell}_{j,k} = \dot{\ell}_{k,j}$. Note that this defines a deformation family of complex structures on C over the base V such that the map $\boldsymbol{v} \in V \mapsto \boldsymbol{\ell}(\boldsymbol{v})$ is a diffeomorphism.

We state that the set $(v_i, v_{j,k} + v_{k,j})$ is a system of coordinates on V in the neighborhood of $0 \in V$. To prove this it is sufficient to show that the linear map $\boldsymbol{v} = (v_i, v_{j,k}, v_{k,j}) \in T_0V \mapsto (v_i, v_{j,k} + v_{k,j})$ is non-degenerate. If it would be not true, then there would exist a nontrivial $\boldsymbol{v} = (v_i, v_{j,k}, v_{k,j}) \in T_0V$ with $v_i = 0$ and $v_{j,k} + v_{k,j} = 0$. Let $\dot{\ell}_i = \dot{\ell}_i(\boldsymbol{v})$, $\dot{\ell}_{j,k} = \dot{\ell}_{j,k}(\boldsymbol{v})$ and $\dot{\ell}_{k,j} = \dot{\ell}_{k,j}(\boldsymbol{v})$ be the corresponding derivatives of length. Then $\dot{\ell}_{j,k} = \dot{\ell}_{k,j}$ and

$$0 < \sum_i \frac{\ell_i}{2\pi} v_i \dot{\ell}_i + \sum_{j<k} \frac{\ell_{jk}}{2\pi} v_{j,k} \dot{\ell}_{j,k} + \sum_{j<k} \frac{\ell_{jk}}{2\pi} v_{k,j} \dot{\ell}_{k,j} = \sum_{j<k} \frac{\ell_{jk}}{2\pi} (v_{j,k} + v_{k,j}) \dot{\ell}_{j,k} = 0.$$

The obtained contradiction leads us to the following conclusion: The functions v_i and $v_{j,k} + v_{k,j}$ define a coordinate system on V equivalent to $\boldsymbol{\ell} = (\ell_i, \ell_{jk})$.

Let us return to the holomorphic deformation family of complex structures on C, defined by complex parameters $\lambda_i = e^{-r_i + i\varphi_i}$ and $\lambda_{jk} = e^{-r_{jk} + i\varphi_{jk}}$. It is easy to see that the Jacobian $\frac{\partial(v_i, v_{j,k} + v_{k,j})}{\partial(r_i, r_{jk})}$ at the point $(r_i, r_{jk}) = 0$ is the identity matrix. This fact proves the statement of the lemma. □

Remark. At this point we give a possible reason why the complex (*i.e.* holomorphic) structure introduced by the complex coordinates $\boldsymbol{\lambda}$ can be regarded as natural. Let C be a complex curve with marked points and nonempty marked boundary. In the case when C is a disk or an annulus assume additionally that at least one inner point of C is marked. The in a neighborhood of every boundary circle γ_i of C we can construct the intrinsic coordinate ζ_i. Take 2 copies C^+ and C^- of C and denote by τ the natural holomorphic map $\tau : C^\pm \to C^\mp$ interchanging the copies. Denote by $\zeta_i^\pm$ the local intrinsic coordinate on $C^\pm$ at boundary circles $\gamma_i^\pm$, both corresponding to γ_i. Now we can glue C^+ and C^- together along every pair of circles (γ_i^+, γ_i^-) by setting $\zeta_i^+ \cdot \zeta_i^- = 1$ as transition relations. We obtain a closed complex curve C^d which admits a natural holomorphic involution $\tau : C^d \to C^d$. For the constructed family $\{C_{\boldsymbol{\lambda}}\}$ the corresponding family $\{C^d_{\boldsymbol{\lambda}}\}$ will be holomorphic. In fact, the statement of *Theorem 2.3* means that $\{C^d_{\boldsymbol{\lambda}}\}$ is a minimal complete family of deformation of C^d in the class of curves with holomorphic involution. This construction of doubling should not be confused with another construction of the *Schottky double* C^{Sch} of C which provides an *antiholomorphic* involution $\tau^{Sch} : C^{Sch} \to C^{Sch}$. We shall use the Schottky double C^{Sch} in *Section 5* considering curves with totally real boundary conditions.

The construction of (holomorphic) double C^d shows how to give an invariant description of holomorphic structure on $\mathbb{T}_\Gamma$. Let C be a smooth complex curve with marked points and marked boundary, and $x \in \mathbb{T}_\Gamma$ the corresponding point on moduli space. Denote by D the divisor of marked points. If curve the C is not closed and C^d is its double with holomorphic involution τ, we denote by $D^d := D + \tau(D)$ the double of D.

Lemma 2.4. *If C is closed, then the tangent space $T_x\mathbb{T}_\Gamma$ is naturally isomorphic to $\mathsf{H}^1(C, \mathcal{O}(TC) \otimes \mathcal{O}(-D))$.*

If C is not closed, then the space $T_x\mathbb{T}_\Gamma$ is naturally isomorphic to the space $\mathsf{H}^1(C^d, \mathcal{O}(TC^d)\otimes\mathcal{O}(-D^d))^\tau$ of τ-invariant elements in $\mathsf{H}^1(C^d, \mathcal{O}(TC^d)\otimes\mathcal{O}(-D^d))$.

In both cases the complex structure on $T_x\mathbb{T}_\Gamma$ induced by local complex coordinates $\boldsymbol{\lambda}$ coincides with those from $\mathsf{H}^1(C, \mathcal{O}(TC)\otimes\mathcal{O}(-D))$ (resp. $\mathsf{H}^1(C^d, \mathcal{O}(TC^d)\otimes\mathcal{O}(-D^d))^{(\tau)}$). In particular, this defines a global complex structure on space $\mathbb{T}_\Gamma$.

Proof. The part concerning closed curves is well-known. In fact, the natural isomorphism $\psi : T_x\mathbb{T}_\Gamma \to \mathsf{H}^1(C, \mathcal{O}(TC)\otimes\mathcal{O}(-D))$ is a Kodaira-Spencer map. Its description is very simple in the introduced local coordinates $\zeta_{j,k}$ on C and $\boldsymbol{\lambda} = (\lambda_{jk})$ on $\mathbb{T}_\Gamma$. Let $C\backslash\{\textit{marked points}\} = \cup C_j$ be the decomposition of C into pants with the graph Γ. For every pants C_j choose an open set $\tilde{C}_j$, containing a closure $\overline{C}_j = C_j \cup \partial C_j$. Without loss of generality we may assume that $\tilde{C}_j$ are chosen not too big, so that the covering $\mathcal{U} := \{\tilde{C}_j\}$ is acyclic for the sheaf $\mathcal{O}(TC)$ and that the local coordinates $\zeta_{j,k}$ are well-defined in the intersections $\tilde{C}_j \cap \tilde{C}_k$. Then vector $v \in T_x\mathbb{T}_\Gamma$ with local representation $v = \sum_{j<k} v_{jk}\frac{\partial}{\partial\lambda_{jk}}$ is mapped by Kodaira-Spencer map ψ to the Čech 1-cohomology class

$$\psi(v) \in \mathsf{H}^1(C, \mathcal{O}(TC)\otimes\mathcal{O}(-D)) \cong \check{\mathsf{H}}^1(\mathcal{U}, \mathcal{O}(TC)\otimes\mathcal{O}(-D)),$$

represented by the 1-cocycle

$$\left(v_{jk}\zeta_{j,k}\frac{\partial}{\partial\zeta_{j,k}}\right) \in \prod_{j<k}\Gamma(\tilde{C}_j\cap\tilde{C}_k, \mathcal{O}(TC)\otimes\mathcal{O}(-D)).$$

For more details see [D-G].

Using this description of Kodaira-Spencer map for closed curves with marked points, it is easy to handle the case of curves with boundary. Let C be a non-compact curve with marked points and with decomposition $C\backslash\{\textit{marked points}\} = \cup_j C_j$. Take its double C^d with the involution τ. Then the decomposition of C induces a τ-invariant decomposition $C^d = \bigcap_j (C_j \cap \tau C_j)$. The corresponding covering $\mathcal{U}^d$ of C^d can be also chosen to be τ-invariant.

The local coordinates ζ_i, corresponding to boundary circles γ_i of C, can be now extended to a both side neighborhood of γ_i in C^d. The coordinates $\zeta_{j,k}$, corresponding to inner circles γ_{jk}, induce local complex coordinates $\zeta^\tau_{j,k} := \zeta_{j,k}\circ\tau$ in $\tau(\tilde{C}_j \cap \tilde{C}_k)$.

Any deformation of the complex structure on C induces a deformation of the complex structure on C^d. This defines a map $\varphi : \mathbb{T}_\Gamma \to \mathbb{T}_{\Gamma^d}$, with Γ^d denoting the graph corresponding to the τ-invariant decomposition of C^d into pants. Using introduced coordinates, we present a tangent vector $v \in T_x\mathbb{T}_\Gamma$ in the form

$$v = \sum_i v_i\frac{\partial}{\partial\lambda_i} + \sum_{j<k} v_{jk}\frac{\partial}{\partial\lambda_{jk}}.$$

Then the composition of the Kodaira-Spencer map ψ^d of C^d with the differential of $\varphi : \mathbb{T}_\Gamma \to \mathbb{T}_{\Gamma^d}$ maps v to

$$\psi^d\circ d\varphi(v) \in \mathsf{H}^1(C^d, \mathcal{O}(TC^d)\otimes\mathcal{O}(-D^d)) \cong \check{\mathsf{H}}^1(\mathcal{U}^d, \mathcal{O}(TC^d)\otimes\mathcal{O}(-D^d)),$$

represented by the Čech 1-cocycle

$$\check{v} := \left(v_i\zeta_i \frac{\partial}{\partial\zeta_i}, v_{jk}\zeta_{j,k}\frac{\partial}{\partial\zeta_{j,k}}, v_{jk}\zeta^\tau_{j,k}\frac{\partial}{\partial\zeta^\tau_{j,k}} \right) \in \prod_i \Gamma(\tilde{C}(i)\cap\tau C(i), \mathcal{O}(TC^d)\otimes\mathcal{O}(-D^d))\times$$

$$\prod_{j<k}\Gamma(\tilde{C}_j\cap\tilde{C}_k, \mathcal{O}(TC^d)\otimes\mathcal{O}(-D^d))\times\prod_{j<k}\Gamma(\tau(\tilde{C}_j\cap\tilde{C}_k), \mathcal{O}(TC^d)\otimes\mathcal{O}(-D^d)),$$

where $C(i)$ denotes the pants of C adjacent to circle γ_i. It is obvious that if all v_i vanish, then this Čech 1-cocycle is τ-invariant. On the other hand, the relation $\zeta_i\cdot(\zeta_i\circ\tau)\equiv\lambda_i = \mathsf{const}$ implies that $\tau_*(\zeta_i\frac{\partial}{\partial\zeta_i}) = -\zeta_i\frac{\partial}{\partial\zeta_i}$. The additional change of sign of the corresponding part of cocycle $\check{v}$ comes from the fact that τ interchange $C(i)$ with $\tau(C(i))$. This shows that $\check{v}$ is τ-invariant and the statement of the lemma follows. □

Now we study the connection between the geometry of $\mathbb{T}_\Gamma$ and the degeneration of complex structures on a real surface Σ with marked points and marked boundary. Let $\Sigma\backslash\{\textit{marked points}\} = \cup_j S_j$ be a decomposition into pants with graph Γ. The Fenchel-Nielsen coordinates on $\mathbb{T}_\Gamma$ define a map $(\boldsymbol{\lambda},\boldsymbol{\vartheta}) : \mathbb{T}_\Gamma \to (\mathbb{R}\times S^1)^{3g-3+m+2b}$, which is a diffeomorphism by *Proposition 1.2*. So, if $\{j_n\}$ is a sequence of complex structures on Σ, its degeneration means that the sequence of Fenchel-Nielsen coordinates of $\{j_n\}$ is not bounded in $(\mathbb{R}\times S^1)^{3g-3+m+2b}$.

One can see that, in fact, we have two types of the degeneration. The first one occurs when the maximum of the length coordinates ℓ_i and ℓ_{jk} of j_n increases infinitely, and the second one is present when minimum of the length coordinates of j_n vanishes. It should be pointed out that for an appropriate sequence one can have both types of degeneration.

Note that by *Proposition 1.2* the Fenchel-Nielsen coordinates of a complex structure j on Σ are defined by a choice of a topological type of decomposition of Σ into pants, encoded in graph the Γ. Thus the introduced notion of degeneration also depends on the choice of Γ. Possibly, the best choice of such decomposition is established by the following statement, proved in [Ab], Ch.II, § 3.3.

Proposition 2.5. *Let C be a complex curve with parametrization $\sigma:\Sigma\to C$. Then*

a) there exists a universal constant $l^*>0$ such that any two geodesic circles γ' and γ'' on C satisfying $\ell(\gamma')<l^*$ and $\ell(\gamma'')<l^*$ are either disjoint or coincide;

b) there exists pants decomposition $C\backslash\{\text{marked points}\} = \cup_j S_j$ such that the lengths of inner boundary circles $\gamma_{jk} = \overline{S}_j\cap\overline{S}_k$ are bounded from above by a constant L which depends only on the topology of Σ and the maximum M of the lengths of the boundary circles of C; moreover, any simple geodesic circle γ on C with $\ell(\gamma)<l^*$ occurs as a boundary circle of some C_j.

Corollary 2.6. *Let C_n be a sequence of nodal curves parametrized by a real surface Σ with uniformly bounded number of components. Suppose that the complex structures of C_n do not degenerate near boundary. Then, passing to a subsequence, one can find a decomposition $\Sigma=\cup_j S_j$ and new parametrizations $\sigma'_n:\Sigma\to C_n$, such that:*

i) the decomposition $\Sigma=\cup_j S_j$ induces a decomposition of every non-exceptional component of C_n into pants, whose boundary circles are geodesics;

ii) the intrinsic length of these geodesics are bounded uniformly in n.

Proof. By *Lemma 2.2*, the intrinsic lengths of boundary circles of non-exceptional components of C_n are bounded uniformly in n. Find a decomposition into pants of every non-exceptional component of C_n satisfying the conditions of part *b)* of *Proposition 2.5*. Let Γ_n denotes the obtained graph of the decomposition of C_n. Since the number of components of C_n is uniformly bounded, we can a subsequence C_{n_k} with the same graph Γ for all n_k.

It follows from the proof in [Ab], that the constant L from part *b)* of *Proposition 2.5* depends continuously on the maximum M of the lengths of the boundary circles of C_n. This implies condition *ii)*. Applying *Proposition 1.2*, we complete the proof. □

3. Apriori estimates.

Let (X, J) be an almost complex manifold. In what follows the tensor J is supposed to be only continuous, *i.e.* of class C^0. Fix some Riemannian metric h on X. All norms and distances will be taken with respect to h. In particular, we have the following

Definition 3.1. *A continuous almost complex structure J is called uniformly continuous on $A \subset X$ with respect to h, if $\|J\|_{L^\infty(A)} < \infty$ and for any $\varepsilon > 0$ there exists $\delta = \delta(J, A, h) > 0$ such that for any $x \in A$ one can find a C^1-diffeomorphism $\varphi : B(x, \delta) \to B(0, \delta)$ from the ball $B(x, \delta) := \{y \in X \ : \ \mathsf{dist}_h(x, y) < \delta\}$ onto the standard ball in $\mathbb{C}^n$ with the standard metric h_{st}, such that*

$$\|J - \varphi^* J_{\mathrm{st}}\|_{L^\infty(B(x,\delta)\cap A)} + \|h - \varphi^* h_{\mathrm{st}}\|_{L^\infty(B(x,\delta)\cap A)} \leqslant \varepsilon.$$

Roughly speaking, this means that on the set A we can $\|\cdot\|_{L^\infty}$-approximate J by an integrable structure in h-metric balls of radius independent of $x \in A$. The function $\mu(J, A, h)$ whose value at $\varepsilon > 0$ is the biggest possible $\delta \leqslant 1$ with the above property is called the *modulus of uniform continuity of J on A*. Note that every continuous almost complex structure J is always uniformly continuous on *relatively compact* subsets $K \Subset X$.

Let J^* be a continuous almost complex structure on X and $A \subset X$ a subset. Assume that J^* is uniformly continuous on A and denote by $\mu_{J^*} = \mu(J^*, A, h)$ the modulus of uniform continuity of J^* on A.

Lemma 3.1. *(First Apriori Estimate). For every p with $2 < p < \infty$ there exists an $\varepsilon_1 = \varepsilon_1(\mu_{J^*}, A, h)$ (independent of p) and $C_p = C(p, \mu_{J^*}, A, h)$, such that for any continuous almost complex structure J with $\|J - J^*\|_{L^\infty(A)} < \varepsilon_1$ and for every J-holomorphic map $u \in C^0 \cap L^{1,2}(\Delta, X)$, satisfying $u(\Delta) \subset A$ and $\|du\|_{L^2(\Delta)} < \varepsilon_1$ one has the estimate*

$$\|du\|_{L^p(\frac{1}{2}\Delta)} \leqslant C_p \cdot \|du\|_{L^2(\Delta)}. \tag{3.1}$$

Proof. *Step 1. First, we prove first inequality* (4.1) *for the case when $A \subset U \subset \mathbb{C}^n$, h is the Euclidean metric, and J^* is the standard complex structure in $\mathbb{C}^n = \mathbb{R}^{2n}$.*

In the Schwarz spaces $\mathcal{S}(\mathbb{C}) = \mathcal{S}(\mathbb{C},\mathbb{C}^n)$ and $\mathcal{S}'(\mathbb{C}) = \mathcal{S}'(\mathbb{C},\mathbb{C}^n)$ we consider the Cauchy-Green operators $\partial = \frac{\partial}{\partial z}$, $\overline{\partial} = \frac{\partial}{\partial \bar z}$, $T = T_{CG} = \frac{1}{2\pi i z} * (\cdot)$ and $\overline{T} = \frac{1}{2\pi i \bar z} * (\cdot)$, where the star $*$ denotes the convolution of distributions. Note that operators T and $\overline{T}$ map $\mathcal{S}$ only to $\mathcal{S}'$ and not in $\mathcal{S}$. Nevertheless one has the following identities in the spaces $\mathcal{S}$ and $L^p(\mathbb{C})$:

$$\overline{\partial}\circ T = T\circ\overline{\partial} = \mathsf{Id} \qquad \text{and} \qquad \partial\circ\overline{T} = \overline{T}\circ\partial = \mathsf{Id}.$$

Recall also that the Calderon-Zygmund inequality states that for any p, $1<p<\infty$, there exists a constant C_p such that for any $f\in L^p(\mathbb{C})$ one has

$$\|(\partial\circ T)(f)\|_{L^p(\mathbb{C})} \leqslant C_p\cdot\|f\|_{L^p(\mathbb{C})} \qquad \text{and} \qquad \|(\overline{\partial}\circ\overline{T})(f)\|_{L^p(\mathbb{C})} \leqslant C_p\cdot\|f\|_{L^p(\mathbb{C})}.$$

This implies that taking any $f\in L^p(\mathbb{C})$ and setting $g := Tf$ (or $g := \overline{T}f$) we get the regularity property $g\in L^{1,p}_{\mathrm{loc}}(\mathbb{C})$ with the estimate $\|dg\|_{L^p(\mathbb{C})} \leqslant (1+C_p)\|f\|_{L^p(\mathbb{C})}$.

Consider now a continuous linear complex structure $J^*(z)$ in the trivial bundle $\mathbb{C}\times\mathbb{R}^{2n}\to\mathbb{C}$. This meant that $J^*(z)$ is a continuous family of endomorphisms $\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ with $J^*(z)^2 = -\mathsf{Id}$. Define an operator $\overline{\partial}_{J^*} : \mathcal{S}'(\mathbb{C},\mathbb{R}^{2n})\to\mathcal{S}'(\mathbb{C},\mathbb{R}^{2n})$ by formula

$$(\overline{\partial}_{J^*}f)(z) = \frac{1}{2}[\partial_x f(z) + J^*(z)\partial_y f(z)].$$

If J is another continuous complex structure in the bundle $\mathbb{C}\times\mathbb{R}^{2n}$ then for $f\in L^p(\mathbb{C},\mathbb{R}^{2n})$ one has the estimate

$$\|(\overline{\partial}_J\circ T - \overline{\partial}_{J^*}\circ T)f\|_{L^p(\mathbb{C})} \leqslant \|J-J^*\|_{L^\infty(\mathbb{C})}\cdot\|d(Tf)\|_{L^p(\mathbb{C})} \leqslant$$

$$\leqslant \|J-J^*\|_{L^\infty(\mathbb{C})}(1+C_p)\|f\|_{L^p(\mathbb{C})}. \tag{3.2}$$

If we take $J^*(z)\equiv J_{\mathrm{st}}$, the standard structure in $\mathbb{C}^n$, then, according to the above remark, $\overline{\partial}_{J^*}\circ T : L^p(\mathbb{C},\mathbb{C}^n)\to L^p(\mathbb{C},\mathbb{C}^n)$ is the identity. From (3.2) we see that if $\|J-J^*\| < \varepsilon_p := \frac{1}{1+C_p}$ then $\overline{\partial}_J\circ T : L^p(\mathbb{C},\mathbb{C}^n)\to L^p(\mathbb{C},\mathbb{C}^n)$ is an isomorphism. Moreover, since $\overline{\partial}_J\circ T = \overline{\partial}_{J^*}\circ T + (\overline{\partial}_J - \overline{\partial}_{J^*})\circ T$, we have

$$(\overline{\partial}_J\circ T)^{-1} = (\mathsf{Id} + (\overline{\partial}_J - \overline{\partial}_{J^*})\circ T)^{-1} = \Sigma_{n=0}^{\infty}(-1)^n[(\overline{\partial}_J - \overline{\partial}_{J^*})\circ T]^n. \tag{3.3}$$

This shows, in particular, that the operator $(\overline{\partial}_J\circ T)^{-1}$ doesn't depend on the choice of $p>1$. Now we shall prove the following

Lemma 3.2. *For any $u\in L^{1,2}(\mathbb{C},\mathbb{R}^{2n})$ with compact support and any continuous J with $\|J-J_{\mathrm{st}}\|_{L^\infty(\mathbb{C},\mathsf{End}(\mathbb{R}^{2n}))} < \varepsilon_p$ the condition $\overline{\partial}_J u\in L^p(\mathbb{C},\mathbb{R}^{2n})$ implies*

$$\|du\|_{L^p(\mathbb{C})} \leqslant C\cdot\|\overline{\partial}_J u\|_{L^p(\mathbb{C})}. \tag{3.4}$$

for some $C = C(p,\|J-J_{\mathrm{st}}\|_{L^\infty(\mathbb{C})})$.

Proof. Put $v = u - T\circ\overline{\partial}_{J_{\mathrm{st}}}u$. Then $\overline{\partial}_{J_{\mathrm{st}}}v = 0$. So v is holomorphic and decreases at infinity. Thus $v=0$, which implies $u = (T\circ\overline{\partial}_{J_{\mathrm{st}}})u$. By Calderon-Zygmund inequality, in order to estimate $\|du\|_{L^p(\mathbb{C})}$ it is sufficient to estimate $\|\overline{\partial}_{J_{\mathrm{st}}}u\|_{L^p(\mathbb{C})}$.

Using $(\overline{\partial}_J \circ T)\circ \overline{\partial}_{J_{\rm st}} u = \overline{\partial}_J u \subset L^p(\mathbb{C})\cap L^2(\mathbb{C})$ and (3.3) we get that $\overline{\partial}_{J_{\rm st}} u \in L^p(\mathbb{C})\cap L^2(\mathbb{C})$ with the estimate

$$\|\overline{\partial}_{J_{\rm st}} u\|_{L^p(\mathbb{C})} \leqslant \Sigma_{n=0}^{\infty} \|(\overline{\partial}_J - \overline{\partial}_{J_{\rm st}})\circ T\|_p^n \cdot \|\overline{\partial}_J u\|_p \leqslant C\cdot \|\overline{\partial}_J u\|_p,$$

which yields (3.4).

To finish *Step 1*, consider a J-holomorphic map $u:\Delta\to(\mathbb{R}^n,J)$, with $u(\Delta)\subset A$ and $\|J-J_{\rm st}\| < \varepsilon_p$. Define a linear complex structure in the bundle $\Delta\times\mathbb{R}^{2n}$ setting $J(z):=J(u(z))$. Then u is a J-holomorphic section of $(\Delta\times\mathbb{R}^{2n},J)$ and $\|J-J_{\rm st}\|_{L^\infty(\Delta)} < \varepsilon_p$. Extend J to $\mathbb{C}\times\mathbb{R}^{2n}$ with the same estimate.

Let ψ be a non-negative cut-off function supported in $\Delta(0,\frac{3}{4})$ which is identically 1 on $\Delta(0,\frac{1}{2})$. Put $u_1 := u\psi$. Then $u_1\in L^{1,2}(\Delta)$ and $\overline{\partial}_J u_1 = u\overline{\partial}_J\psi \in L^p(\mathbb{C})$ with $\|\overline{\partial}_J u_1\|_{L^p(\Delta)} = \|u\overline{\partial}_J\psi\|_{L^p(\Delta)} \leqslant C\|du\|_{L^2(\Delta)}$. Here we use the Sobolev imbedding $L^{1,2}(\Delta,\mathbb{C})\to L^p(\Delta,\mathbb{C})$, $p<\infty$. Now (3.4) applies to get the estimate of *Step 1.*

Using the Sobolev imbedding $L^{1,p}\subset C^{1-\frac{2}{p}}$ and obvious properties of L^p-norms by dilations, one derives easily from *Step 1* the following property

Step 2. Fix $2<p<\infty$. *There exists* $\varepsilon_2=\varepsilon_2(\mu_{J^*},A,h)>0$ *such that for any* u *and* J *as in Lemma 3.1 with* $\mathsf{diam}(u(\Delta(x,r)))<\varepsilon_2$ *one has the estimate*

$$\mathsf{diam}(u(\Delta(x,\tfrac{r}{2}))) \leqslant C_1 r^{1-\frac{2}{p}}\cdot\|du\|_{L^p(\Delta(x,r/2))} \leqslant C_2 r^{1-\frac{2}{p}}\cdot\|du\|_{L^2(\Delta(x,r))}. \tag{3.5}$$

for any disc $\Delta(x,r)\subset\Delta$.

Now consider the function

$$\alpha(r):=\begin{cases} 1 & \text{if} \quad r\leqslant 1/2\\ 3-4r & \text{if} \quad 1/2\leqslant r\leqslant 3/4\\ 0 & \text{if} \quad 3/4\leqslant r\end{cases}$$

For $x\in\Delta$ set

$$f(x):=\max\left\{t\in[0,\tfrac{1}{8}]\ :\ \mathsf{diam}(u(\overline{\Delta}(x,t\cdot\alpha(|x|))))\leqslant\varepsilon_2\right\}.$$

Clearly, f is continuous and $f\equiv\frac{1}{8}$ if $\frac{3}{4}\leqslant|x|<1$.

Step 3. $f(x)\equiv\frac{1}{8}$.

Suppose that there is an x_0 with $f(x_0)=\min\{f(x)\ :\ x\in\Delta\}<\frac{1}{8}$. It is clear that $f(x_0)>0$.

Take the disk $\Delta(x_0,a)$ with $a:=f(x_0)\alpha(|x_0|)$. Note that

$$\mathsf{diam}(u(\Delta(x_0,a)))=\varepsilon_2. \tag{3.6}$$

Using the Sobolev embedding $L^{1,4}(\Delta)\subset C^{0,\frac{1}{2}}(\Delta)$, estimate (3.5), and relation (3.6), we obtain that $\mathsf{diam}(u(\Delta(x_0,\frac{a}{2})))\leqslant C\cdot\|du\|_{L^2(\Delta(x_0,a))}$. Take a point $x_1\in\Delta(x_0,a)$ with $|x_1-x_0|\leqslant\frac{3}{4}a$. Since $f(x_0)=\frac{a}{\alpha(|x_0|)}$ is the minimum of f, we have that $f(x_1)\geqslant\frac{a}{\alpha(|x_0|)}$ and, thus, $f(x_1)\alpha(|x_0|)\geqslant a$. At the same time $\alpha(|x_1|)\geqslant\alpha(|x_0|)-3a$, so $f(x_1)\alpha(|x_1|)\geqslant a-3a^2\geqslant\frac{a}{2}$ because $a\leqslant\frac{1}{8}$. This means that $\mathsf{diam}(u(\Delta(x_1,\frac{a}{2})))\leqslant\varepsilon_2$ and so $\mathsf{diam}(u(\Delta(x_1,\frac{a}{4})))\leqslant C\cdot\|du\|_{L^2(\Delta)}$. So $\mathsf{diam}(u(\Delta(x_0,a)))\leqslant 4C\cdot||du||_{L^2(\Delta)}$.

If ε is taken smaller than $\frac{\varepsilon_2}{4C}$ then we obtain a contradiction with (3.4). *Step 3* is completed.

This means that $\mathsf{diam}(u(\Delta(x,\frac{1}{8}))) \leqslant \varepsilon_2$ for any $x \in \Delta(0,\frac{1}{2})$. So *Step 2* with $r = \frac{1}{8}$ gives us the assertion of *Lemma 3.1*. $\square$

This lemma can be used to prove that a J-holomorphic map $u : \Delta \to (X,J)$ is $L^{1,p}$-smooth for any $p < \infty$, provided J is continuous. To show this, we note that $u_\varepsilon(z) := u(\varepsilon z)$ is also J-holomorphic and $\|du_\varepsilon\|_{L^2(\Delta(0,1))} = \|du\|_{L^2(\Delta(0,\varepsilon))}$. For ε small enough we obtain $\|du_\varepsilon\|_{L^2(\Delta(0,1))} = \|du\|_{L^2(\Delta(0,\varepsilon))} < \varepsilon_1$, where ε_1 is defined in from *Lemma 3.1*. Now, estimate (3.1) gives us the $L^{1,p}$-continuity of u_ε and thus of u in the neighborhood of zero. Another immediate consequence of the First Apriori Estimate (3.1) is the following

Corollary 3.3. *Let X be a manifold, h some metric, and $\{J_n\}$ a sequence of continuous almost complex structures on X such that $J_n \longrightarrow J$ in C^0-topology on X. Let $A \subset X$ be a closed h-complete subset, such that J is uniformly continuous on A w.r.t. h.*

Let $u_n \in C^0 \cap L^{1,2}_{\mathrm{loc}}(\Delta,X)$ be a sequence of J_n-holomorphic maps such that $u_n(\Delta) \subset A$, $\|du_n\|_{L^2(\Delta)} \leqslant \varepsilon_1$, and $u_n(0)$ is bounded in X. Then there exists a subsequence $\{u_{n_k}\}$ which $L^{1,p}_{\mathrm{loc}}$-converges to a J-holomorphic map u_∞ for all $p < \infty$.

In particular, for any $K \Subset \Delta$ norms $\|du_{n_k}\|_{L^2(K)}$ tend to $\|du_\infty\|_{L^2(K)}$.

Proof. First Apriori Estimate of *Lemma 3.1* with the Sobolev imbedding $L^{1,p}(\Delta) \subset C^{1-\frac{2}{p}}(\Delta)$ implies that for every $r < 1$ the sequence of the sets $\{u_n(\Delta(0,r))\}$ is uniformly bounded in X. Consequently, $u_n(\Delta(0,r)) \subset K_r$ for some relatively compact subset $K_r \Subset X$ independent of n. This implies the existence of a subsequence $\{u_{n_k}\}$ which converges to u_∞ in $C^\alpha(\Delta,X)$ for any $\alpha < 1$.

To show the (strong) $L^{1,p}_{\mathrm{loc}}(\Delta)$-convergence of $\{u_{n_k}\}$, take any $x \in \Delta$. Then we can find $r > 0$, such that all images $u_{n_k}(\overline{\Delta}(x,r))$ lie in some chart $U \subset X$. Moreover, we may assume that U is a domain in $\mathbb{C}^n$, such that $\|J - J_{\mathrm{st}}\|_{L^\infty(U)} \leqslant \varepsilon_1$. Take $\varphi \in C^\infty_0(\Delta(x,r))$ with $\varphi\big|_{\Delta(x,r/2)} \equiv 1$. Then

$$\overline{\partial}_{J_{n_k}}(\varphi u_{n_k}) = \partial_x(\varphi u_{n_k}) + J_{n_k}(u_{n_k})\partial_y(\varphi u_{n_k}) = (\partial_x\varphi + J_{n_k}(u_{n_k})\partial_y\varphi)u_{n_k},$$

which is C^0-bounded and thus L^p-convergent for any $p < \infty$. The estimate (3.4) for $\overline{\partial}_{J_{n_k}}$-operator gives us the $L^{1,p}$-convergence of $\{u_{n_k}\}$ on $\Delta(x,r/2)$. $\square$

Definition 3.2. *Define a cylinder $Z(a,b)$ by $Z(a,b) := S^1 \times [a,b]$, equiping it with coordinates $\theta \in [0,2\pi]$, $t \in [a,b]$, with metric $ds^2 = d\theta^2 + dt^2$ and the complex structure $J_{\mathrm{st}}(\frac{\partial}{\partial\theta}) = \frac{\partial}{\partial t}$. Denote $Z_i := Z(i-1,1) = S^1 \times [i-1,i]$.*

Let J^* be some continuous almost complex structure on X and A a subset of X, such that J^* is uniformly continuous on A. Let μ_{J^*} denote the modulus of uniform continuity of J^* on A.

Lemma 3.4. *(Second Apriori Estimate). There exist constants $\gamma \in]0,1[$ and $\varepsilon_2 = \varepsilon_2(\mu_{J^*},A,h) > 0$ such that for any J with $\|J - J^*\| < \varepsilon_2$ and every J-holomorphic map $u : Z(0,5) \to X$ with $u(Z(0,5)) \subset A$ the condition $\|du\|_{L^2(Z_i)} < \varepsilon_2$ for $i = 1,\dots,5$ implies*

$$\|du\|^2_{L^2(Z_3)} \leqslant \frac{\gamma}{2}\big(\|du\|^2_{L^2(Z_2)} + \|du\|^2_{L^2(Z_4)}\big). \tag{3.7}$$

Proof. Take $\varepsilon_2>0$ small enough, such that $\mu_{J^*}(\varepsilon_2)<\varepsilon_1$, where ε_1 is the constant from *Lemma 3.1*. Then for any $A'\subset A$ the condition $\mathsf{diam}(A')\leqslant\varepsilon_2$ implies that $\mathsf{osc}(J^*,A')\leqslant\varepsilon_1$. Due to *Lemma 3.1*, we may assume that $u(Z_i)\subset B$ for $i=2,3,4$, where B is a small ball in $\mathbb{R}^{2n}=\mathbb{C}^2$ with the structure J_{st}. Moreover, we may assume that $\|J^*-J_{\mathrm{st}}\|_{L^\infty(B)}\leqslant\varepsilon_1$.

Find $v\in C^0\cap L^{1,2}(Z(1,4),\mathbb{C}^n)$ such that $\overline{\partial}_{J_{\mathrm{st}}}v=0$ and $\|du-dv\|_{L^2(Z(1,4))}$ is minimal. We have

$$\|\overline{\partial}_{J_{\mathrm{st}}}(u-v)\|_{L^2(Z_i)}=\|(J_{\mathrm{st}}-J(u))\partial_y u\|_{L^2(Z_i)}\leqslant\|J_{\mathrm{st}}-J\|_{L^\infty(B)}\|du\|_{L^2(Z_i)}.$$

So for $i=2,3,4$ we get

$$\|du-dv\|_{L^2(Z_i)}\leqslant C\|J_{\mathrm{st}}-J\|_{L^\infty(B)}\|du\|_{L^2(Z(1,4))}. \tag{3.8}$$

Now let us check the inequality (3.7) for v. Write $v(z)=\Sigma_{k=-\infty}^{\infty}v_k e^{k(t+i\theta)}$. Then $\|dv\|^2_{L^2(S\times\{t\})}=4\pi\Sigma_{k=-\infty}^{\infty}k^2|v_k|^2e^{2kt}$. Since obviously

$$\int_2^3 e^{2kt}\leqslant\frac{\gamma_1}{2}\left(\int_1^2 e^{2kt}dt+\int_3^4 e^{2kt}dt\right)$$

for all $k\neq 0$ with $\gamma_1=\frac{2}{e^2}$, one gets the required estimate for all holomorphic v.

Using (3.8) with $\|J_{\mathrm{st}}-J\|_{L^\infty}$ sufficiently small, we conclude that the estimate (3.7) holds for u with appropriate $\gamma>\gamma_1$. □

Corollary 3.5. *Let X, h, J^*, A, and the constants ε_2 and γ be as in Lemma 3.4. Suppose that J is a continuous almost complex structure on X with $\|J-J^*\|_{L^\infty(A)}<\varepsilon_2$ and $u\in C^0\cap L^{1,2}(Z(0,l),X)$ a J-holomorphic map, such that $u(Z)\subset A$ and $\|du\|_{L^2(Z_i)}<\varepsilon_2$ for any $i=1,\dots,l$. Let $\lambda>1$ be (the uniquely defined) real number with $\lambda=\frac{\gamma}{2}(\lambda^2+1)$.*

Then for $2\leqslant k\leqslant l-1$ one has

$$\|du\|^2_{L^2(Z_k)}\leqslant\lambda^{-(k-2)}\cdot\|du\|^2_{L^2(Z_2)}+\lambda^{-(l-1-k)}\cdot\|du\|^2_{L^2(Z_{n-1})}. \tag{3.9}$$

Proof. The definition of λ implies that for any a_+ and a_- the sequence $y_k:=a_+\lambda^k+a_-\lambda^{-k}$ satisfies the recurrent relation $y_k=\frac{\gamma}{2}(y_{k-1}+y_{k+1})$. In particular, so does the sequence

$$A_k:=\frac{\lambda^{-(k-2)}-\lambda^{6-2l+k-2}}{1-\lambda^{6-2l}}\|du\|^2_{L^2(Z_2)}+\frac{\lambda^{-(l-1-k)}-\lambda^{6-2l+l-1-k}}{1-\lambda^{6-2l}}\|du\|^2_{L^2(Z_{l-1})},$$

which is detemined by the values $A_2=\|du\|^2_{L^2(Z_2)}$ and $A_{l-1}=\|du\|^2_{L^2(Z_{l-1})}$.

We claim that for $2\leqslant k\leqslant l-1$ one has the estimate $\|du\|^2_{L^2(Z_k)}\leqslant A_k$, which is obviously stronger than (3.9). Suppose that there exists a k_0, such that $2\leqslant k_0\leqslant l-1$ and $\|du\|^2_{L^2(Z_{k_0})}>A_{k_0}$. Choose k_0 so that the difference $\|du\|^2_{L^2(Z_{k_0})}-A_{k_0}$

is maximal. By *Lemma 3.3* and by our recurrent definition of A_k we have that $2 < k_0 < l-1$ and

$$\|du\|^2_{L^2(Z_{k_0})} - A_{k_0} \leqslant \frac{\gamma}{2}(\|du\|^2_{L^2(Z_{k_0+1})} - A_{k_0+1} + \|du\|^2_{L^2(Z_{k_0-1})} - A_{k_0-1}) \leqslant$$

$$\leqslant \frac{\gamma}{2} 2(\|du\|^2_{L^2(Z_{k_0})} - A_{k_0})$$

The second inequality follows from the fact that $\|du\|^2_{L^2(Z_{k_0})} - A_{k_0}$ is maximal. This gives a contradiction. □

An immediate corollary of this estimate is the following improvement of Sacks-Uhlenbeck theorem about removability of a point singularity, see [S-U] and [G].

Corollary 3.6. *(Removal of point singularities). Let X be a manifold with a Riemannian metric h, J a continuous almost complex structure, and $u : (\check{\Delta}, J_{\mathrm{st}}) \to (X, J)$ a pseudoholomorphic map from the punctured disk. Suppose that*

i) J is uniformly continuous on $A := u(\check{\Delta})$ w.r.t. h and the closure of A is h-complete;

ii) there exists i_0, such that for all annuli $R_i := \{z \in \mathbb{C} : \frac{1}{e^{i+1}} \leqslant |z| \leqslant \frac{1}{e^i}\}$ with $i \geqslant i_0$ one has $\|du\|^2_{L^2(R_i)} \leqslant \varepsilon_2$, where ε_2 is defined in Lemma 3.3.

Then u extends to the origin.

Condition *i)* is automatically satisfied if $A = u(\check{\Delta})$ is relatively compact in X. Condition *ii)* of "slow growth" is clearly weaker than just the boundedness of the area, see e.g. [S-U], [G]. It is sufficient to have $\lim_{i \longrightarrow \infty} \|du\|^2_{L^2(R_i)} = 0$, whereas boundedness of the area means $\sum_{i=1}^{\infty} \|du\|^2_{L^2(R_i)} < \infty$.

Proof. The exponential map $\exp(t, \theta) := e^{-t+i\theta}$ defines a biholomorphism between the infinite cylinder $Z(0, \infty)$ and the punctured disk $\check{\Delta}$, identifying every annulus R_i with the cylinder $Z(i, i+1)$. Applying *Corollary 3.5* to the map $u \circ \exp$ on cylinders $Z(i_0, l)$ and setting $l \longrightarrow \infty$, we get the estimate

$$\|du\|^2_{L^2(R_i)} \leqslant \lambda^{-(i-i_0)} \cdot \|du\|^2_{L^2(R_{i_0})}, \qquad i > i_0.$$

Using this and *Lemma 3.1* we conclude that $\operatorname{diam}(u(R_i)) \leqslant C \cdot \lambda^{-i/2}$ for $i > i_0$. Since $\sum \lambda^{-i/2} < \infty$, u extends continuously into $0 \in \Delta$. □

In the proof of the compactness theorem we shall use the following corollary from *Lemma 3.3*. Let X be a manifold with a Riemannian metric h, J a continuous almost complex structure on X, $A \subset X$ a closed h-complete subset, such that J is h-uniformly continuous on A. Furthermore, let $\{J_n\}$ be a sequence of almost complex structures uniformly converging to J, $\{l_n\}$ a sequence of integers with $l_n \to \infty$, and $u_n : Z(0, l_n) \to X$ a sequence of J_n-holomorphic maps.

Lemma 3.7. *Suppose that $u_n(Z(0, l_n)) \subset A$ and $\|du_n\|_{L^2(Z_i)} \leqslant \varepsilon_2$ for all n and $i \leqslant l_n$. Take a sequence $k_n \to \infty$ such that $k_n < l_n - k_n \to \infty$. Then:*

1) $\|du_n\|_{L^2(Z(k_n, l_n - k_n))} \to 0$ and $\operatorname{diam}\big(u_n(Z(k_n, l_n - k_n))\big) \to 0$;

2) if, in addition, all images $u_n(Z(0, l_n))$ are contained in some bounded subset of X, then there is a subsequence $\{u_n\}$, still denoted $\{u_n\}$, such that both $u_n|_{Z(0,k_n)}$ and $u_n|_{Z(k_n,l_n)}$ converge in $L^{1,p}$-topology on compact subsets in $\check{\Delta} \cong Z(0, +\infty)$ to

J-holomorphic maps $u_\infty^+ : \check\Delta \to X$ and $u_\infty^- : \check\Delta \to X$. Moreover, both u_∞^+ and u_∞^- extend to the origin and $u_\infty^+(0) = u_\infty^-(0)$.

Remarks. 1. The punctured disk $\check\Delta$ with the standard structure $J_\Delta \frac{\partial}{\partial r} = \frac{1}{r}\frac{\partial}{\partial\theta}$ is isomorphic to $Z(0,\infty)$ with the structure $J_Z \frac{\partial}{\partial t} = -\frac{\partial}{\partial\theta}$ under a biholomorphism $(\theta,t) \mapsto e^{-t+i\theta}$. Thus statement (2) of this corollary is meaningful.
2. *Lemma 3.6* describes explicitly how the sequence of J_n-holomorphic maps of the cylinders of growing conformal radii converges to a J-holomorphic map of the standard node.

Lemma 3.8. *There is an $\varepsilon_3 = \varepsilon_3(\mu_{J_\infty}, A, h)$ such that for any continuous almost-complex structure J on X with $\|J - J_\infty\|_{L^\infty} \leqslant \varepsilon_3$ and any non-constant J-holomorphic sphere $u : \mathbb{CP}^1 \to X$, $u(\mathbb{CP}^1) \subset A$ one has the inequalities*

$$\mathsf{area}\,(u(\mathbb{CP}^1)) \geqslant \varepsilon_3 \qquad \textit{and} \qquad \mathsf{diam}(u(\mathbb{CP}^1)) \geqslant \varepsilon_3.$$

Proof. Let ε_1 be the constant from *Lemma 3.1*. Suppose that $\mathsf{area}\, u(\mathbb{CP}^1) = \|du\|^2_{L^2(\mathbb{CP}^2)} \leqslant \varepsilon_1^2$. Cover $\mathbb{CP}^1$ by two disks Δ_1 and Δ_2. By (3.1) and Sobolev imbedding $L^{1,p} \subset C^{0,1-\frac{2}{p}}$ we obtain that $\mathsf{diam}(u(\Delta_1))$ and $\mathsf{diam}(u(\Delta_2))$ are smaller than $const \cdot \varepsilon_1$. Thus the diameter of the image of the sphere is smaller than $const \cdot \varepsilon_1$.

So we can suppose that the image $u(S^2)$ is contained in the coordinate chart, *i.e.* in a subdomain in $\mathbb{C}^n$, and the structures J and J_∞ are L^∞-close to a standard one. Consider now $u : S^2 \to U \subset \mathbb{C}^n$ as a solution of the linear equation

$$\partial_x v(z) + J(u(z)) \cdot \partial_y v(z) = 0 \tag{3.10}$$

on the sphere. The operator $\overline\partial_J(v) = d_x v(z) + J(u(z)) \cdot \partial_y v(z)$ acts from $L^{1,p}(S^2, \mathbb{C}^n)$ to $L^p(S^2, \mathbb{C}^n)$ and is a small perturbation of the standard $\overline\partial$-operator. Note that the standard $\overline\partial$ is surjective and Fredholm. Thus small perturbations are also surjective and Fredholm, having the kernel of the same dimension. But the kernel of $\overline\partial$ consists of constant functions. Since all constants are in the kernel of (3.10), our u should be a constant map.

We have proved that if the area or a diameter of J-holomorphic map is sufficiently small then this map is constant. □

Remark. The same statement is true for the curves of arbitrary genus g. In that case, in addition to the estimate (3.1), one should also use the estimate (3.7). This yields the existence of an ε which depends on g (and, of course, on X, J, and K), but not on the complex structure on the parameterizing surface.

4. Compactness for curves with free boundary

In this section we give a proof of the Gromov compactness theorem for the curves with boundaries of fixed finite topological type and without boundary conditions on maps. The case of closed curves is obviously included in this one.

Throughout this section we assume that the following setting holds.

Let X be a manifold with a Riemannian metric h, J_∞ a continuous almost complex structure on X, $A \subset X$ an h-complete subset, $\{C_n\}$ a sequence of nodal curves parametrized by a real surface Σ with parametrizations $\delta_n : \Sigma \to C_n$, and $u_n : (C_n, j_n) \to (X, J_n)$ a sequence of pseudoholomorphic maps. Further, J_∞ is h-uniformly continuous on A, J_n are also continuous and converge to J_∞, h-uniformly on A, $u_n(C_n) \subset A$ for all n.

Let us explain the main idea of the proof of *Theorem 1.1.* The Gromov topology on the space of stable curves over X is introduced in order to recover breaking of "strong" (*i.e.* $L^{1,p}$-type) convergence of a sequence (C_n, u_n) of pseudoholomorphic curves of bounded area. The are two reasons for this. The first one is that a sequence of (say, smooth) curves C_n could diverge in an appropriate moduli space and the second one is a phenomenon of "bubbling". In both cases one has to do with appearance of new nodes, *i.e.* with certain degeneration of complex structure on curves. The "model" situation of *Lemma 3.6* describes a convergence of "long cylinders" $u_n : Z(0, l_n) \to X$, $l_n \longrightarrow \infty$, to a node $u_\infty : \mathcal{A}_0 \to X$. In our proof we cover curves C_n by pieces, which are either "long cylinders" converging to nodes or have the property that complex structures and maps "strongly" converge. Here the "strong" convergence means the usual one, *i.e.* w.r.t. the C^∞-topology for complex structures, and w.r.t. the $L^{1,p}$-topology with some $p > 2$ for maps. In fact, the strong convergence of maps is equivalent to the uniform one, *i.e.* w.r.t. the C^0-topology, and implies further regularity in the case when J_n and J_∞ have more smoothness. One consequence of this is that we remain is the category of nodal curves. Another one is that we treat degeneration of complex structure on C_n and the "bubbling" phenomenon in a uniform framework of "long cylinders".

For the proof we need some additional results.

Lemma 4.1. *For any $R > 1$ there exists an $a^+ = a^+(R) > 0$ with the following property. For any cylinder $Z = Z(0, l)$ with $0 < l \leqslant +\infty$ and any annulus $A \subset Z(0, l)$, which is adjacent to $\partial_0 Z = S^1 \times \{0\}$ and has conformal radius R, one has $Z(0, a^+) \subset A$.*

Proof. Without loss of generality we may assume that $l = +\infty$ and identify Z with the punctured disk $\check{\Delta}$ via the exponential map $(-t + i\theta) \mapsto e^{-t+i\theta}$, such that $\partial_0 Z$ is mapped onto $S^1 = \partial\Delta$.

Suppose that the statement is false. Then there would exist holomorphic imbeddings $f_n : A(1, R) \to \check{\Delta}$ and points $a_n \in \Delta \backslash f_n(A(1, R))$, such that $f_n(A(1, R))$ are adjacent to $\partial\Delta$ and $a_n \longrightarrow a \in \partial\Delta$. Passing to a subsequence, we may assume that $\{f_n\}$ converges uniformly on compact subsets in $A(1, R)$ to a holomorphic map $f : A(1, R) \to \Delta$.

If f is not constant, then $f(A(1, R))$ must contain some annulus $\{b < |z| < 1\}$ with $b < 1$. But then $\{\sqrt{b} < |z| < 1\} \subset f_n(A(1, R))$ for $n >> 1$, which is a contradiction.

If f is constant, then the diameter of images of the middle circle $\gamma := \{|z| = \sqrt{R}\} \subset A(1, R)$ must converge to 0. But $\mathsf{diam}(f_n(\gamma)) \geqslant \mathsf{dist}(0, a_n) \longrightarrow 1$. The obtained contradiction finishes the proof. $\square$

For the proof of *Theorem 1.1* we need a special covering of Σ which will be constructed in the following theorem.

Theorem 4.2. *Under the conditions of Theorem 1.1, after passing to a subsequence, there exist a finite covering $\mathcal{V}$ of Σ by open sets V_α and parametrizations $\sigma_n : \Sigma \to C_n$ such that:*

(a) all V_α are either disks, or annuli, or pants;

(b) for any boundary circle γ_i of Σ there is some annulus V_α adjacent to γ_i;

(c) $\sigma_n^* j_n\big|_{V_\alpha}$ doesn't depend on n if V_α is a disk, pants, or an annulus adjacent to a boundary circle of Σ;

(d) all non-empty intersections $V_\alpha \cap V_\beta$ are annuli, where the structures $\sigma^* j_n$ are independent of n;

(e) if a is a node of C_n and $\gamma_a^n = \sigma_n^{-1}(a)$ the corresponding circle, then $\gamma_a^n = \gamma_a$ doesn't depend on n, is contained in some annulus V_α, containing only one such "contracting" circle for any n; moreover, the structures $\sigma_n^* j_n\big|_{V_\alpha \setminus \gamma_a}$ are independent of n;

(f) if V_α is an annulus and $\sigma_n(V_\alpha)$ are not nodes, then the conformal radii of $\sigma_n(V_\alpha)$ converge to some positive $R_\alpha^\infty > 1$ or to $+\infty$.

(g) if for initial parametrizations δ_n and fixed annuli A_i, each adjacent to the boundary circle γ_i of Σ, the structures $\delta_n^* j_n\big|_{A_i}$ do not depend on n A_i, then the new parametrizations σ_{n_k} can be taken equal to δ_{n_k} on some subannuli $A_i' \subset A_i$ also adjacent to γ_i.

Proof. We shall prove the properties *(a)–(f)*. The property *(g)* will follow from *Lemma 4.3* below.

There are 4 cases where the existence of such a covering is obvious. If all C_n are disks or annuli without nodal points, there is nothing to prove. In the third case each C_n is a sphere, and we cover it by 2 disks.

In the forth case each C_n is a torus without marked points. Then Any complex torus can be represented in the form $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ with $|\mathsf{Re}\,\tau| \leqslant \frac{1}{2}$ and $\mathsf{Im}\,\tau > \frac{1}{2}$. Considering the map $z \in \mathbb{C} \mapsto e^{2\pi i z} \in \check{\mathbb{C}} := \mathbb{C}\backslash\{0\}$, we represent (T^2, j) as the quotient $\mathbb{C} \;/\; \{z \sim \lambda^2 z\}$ with $\lambda = e^{\pi i \tau}$, so that $|\lambda| < e^{-\pi/2} < \frac{1}{3}$. The annuli $\{\frac{|\lambda|}{2} < |z| < 1\}$ and $\{\frac{|\lambda|^2}{2} < |z| < |\lambda|\}$ form the needed covering.

In all remaining cases we start with constructing of appropriate graphs Γ_n associated with some decomposition of C_n into pants. *Lemma 3.4* and non-degeneration of the complex structure j_n on C_n shows that lengths of all boundary circles of all non-exceptional components $C_{n,i}$ of C_n are uniformly bounded from above. At this point we make the following

Remark. The Collar Lemma from [Ab], Ch.II, § 3.3 yields the existence of the universal constant l^* such that for any simple geodesic circles γ' and γ'' on $C_{n,i}$ the conditions $\ell(\gamma') < l^*$ and $\ell(\gamma'') < l^*$ imply $\gamma' \cap \gamma'' = \varnothing$. We shall call geodesic circles γ with $\ell(\gamma) < l^*$ *short geodesics*.

The fact that (C_n, u_n) are pseudoholomorphic and of bounded area shows that C_n have uniformly bounded number of components. Indeed, the number of exceptional components, which are spheres and disks, is bounded by the energy (see *Lemma 3.8*), whereas the number of boundary circles of Σ. Further, the operation of contracting a circle on Σ to a nodal point either dininish genus of some component of C_n, or increase the number of components. Thus, the number of nodal points on C_n and the total number of marked points on its components are also

uniformly bounded. This implies that the number of possible topological types of components is finite.

In this situation the Teichmüller theory (see [Ab], Ch.II, § 3.3) states the existence of decomposition of every non-exceptional component $C_{n,i}\backslash\{marked\ points\}$ into pants with the following properties:

i) every short geodesic is a boundary circle of some pants of the decomposition;

ii) the intrinsic length of every boundary circle is bounded from above by a (uniform) constant depending only on an upper bound of lengths of boundary circles and possible topological types of $C_{n,i}\backslash\{marked\ points\}$.

Having decomposed all $C_{n,i}\backslash\{marked\ points\}$ into pants, we associate with every curve C_n its graph Γ_n. As it was noted above, the number of vertices and edges of Γ_n is uniformly bounded. Thus after passing to a subsequence, we can assume that all Γ_n are isomorphic to each other (as marked graphs). Denote this graph by Γ. Now, the parametrizations $\sigma_n : \Sigma \to C_n$ can be found in such a way that the decompositions of $C_{n,i}\backslash\{marked\ points\}$ into pants define the same set $\boldsymbol{\gamma} = \{\gamma_\alpha\}$ of circles on Σ and induce the same decomposition $S\backslash \cup_\alpha \gamma_\alpha = \cup_j S_j$ with the graph Γ.

By our construction of the graph Γ, each edge of Γ corresponds either to a circle in Σ contracted by every parametrization σ_n to a nodal point, or to a circle mapped by every σ_n onto a geodesic circle separating two pants. Furthermore, each tail of Γ corresponds to a boundary circle of Σ. Thus we shall use the same notation γ_α for an edge or a tail of Γ and for the corresponding circle on Σ. If γ_α is a boundary circle of some pants S_j, then the intrinsic length $\ell_{n,\alpha} = \ell_n(\gamma_\alpha)$ of $\sigma_n(\gamma_\alpha)$ is well defined. This happens in the following two cases:

a) γ_α separates two pants, or else

b) γ_α is a boundary circle Σ and for any n the irreducible component of C_n adjacent to $\sigma_n(\gamma_\alpha)$ is not a disk with a single nodal point. Note that appearance of these two cases is independent of n.

By our choice of γ_α, the lengths $\ell_n(\gamma_\alpha)$ are uniformly bounded from above. Passing to a subsequence, we may assume that for any fixed α the sequence $\{\ell_{n,\alpha}\}$ converges to $\ell_{\infty,\alpha}$.

As one can expect, the condition $\ell_{n,\alpha} \longrightarrow 0$ means that the circle γ_α is shrunk to a nodal point on the limit curve. We shall prove the statement of the theorem by induction in the number N of those circles γ_α for which $\ell_{\infty,\alpha} = 0$.

The case $N = 0$, when there are no such circles, is easy. Passing to a subsequence, we may assume that the Fenchel-Nielsen coordinates $(\boldsymbol{\ell}_n, \boldsymbol{\vartheta}_n)$ of any non-exceptional component $C_{n,i}$ of C converge to the Fenchel-Nielsen coordinates $(\boldsymbol{\ell}_\infty, \boldsymbol{\vartheta}_\infty)$ of some smooth curve $C_{\infty,i}$ with marked points. Gluing together appropriate pairs of marked points we obtain a nodal curve C_∞, which admits a suitable parametrization $\sigma_\infty : \Sigma \to C_\infty$ and has the same graph Γ. *Lemma 3.5* shows that for $n >> 1$ the curves C_n can be obtained from C_∞ by deformation of the transition functions for the intrinsic local coordinates on non-exceptional components of C_∞. Note that such a deformation can be realized as a deformation of operator j_∞ of complex structure on C_∞, localized in small neighborhoods of circles $\sigma_\infty(\gamma_\alpha)$, see Fig. 6. In the case when γ_α is a boundary circle, we may additionally assume that the annulus, where j_n changes, lies away from γ_α. Now the existence of the covering with desired properties is obvious.

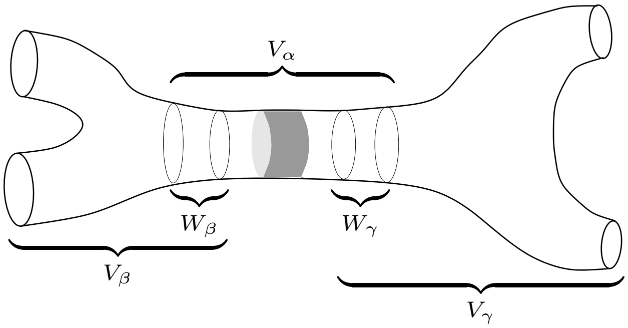

Fig. 6.
V_α and V_β represent elements of the covering where the complex structure is constant. The change of complex structure is done in the painted part of V_γ. $W_\beta := V_\alpha \cap V_\beta$ and $W_\gamma := V_\alpha \cap V_\gamma$ represent the annuli with the constant complex structure.

Let us consider now the general case when the number N of "shrinking circles" is not zero. Take a circle γ_α with $\ell_\infty(\gamma_\alpha) = 0$. Let S_j be pants adjacent to γ_α. Consider the intrinsic coordinates ρ_α and θ_α at $\sigma_n(\gamma_\alpha)$ and the annuli

$$
\begin{aligned}
A_{n,\alpha,j} &:= \left\{ (\rho_\alpha, \theta_\alpha) \in \sigma_n(S_j) \;:\; 0 \leqslant \rho_\alpha \leqslant \frac{\pi^2}{\ell_{n,\alpha}} - \frac{2\pi}{a^*} \right\} \\
A^-_{n,\alpha,j} &:= \left\{ (\rho_\alpha, \theta_\alpha) \in \sigma_n(S_j) \;:\; 0 \leqslant \rho_\alpha \leqslant \frac{\pi^2}{\ell_{n,\alpha}} - \frac{2\pi}{a^*} - 1 \right\},
\end{aligned}
$$

adjacent to $\sigma_n(\gamma_\alpha)$. Note that $\frac{\pi^2}{\ell_{n,\alpha}} - \frac{2\pi}{a^*}$ (resp. $\frac{\pi^2}{\ell_{n,\alpha}} - \frac{2\pi}{a^*} - 1$) is the logarithm of the conformal radius of $A_{n,\alpha,j}$ (resp. of $A^-_{n,\alpha,j}$). Consequently, we can use *Lemma 2.2* to shows that these annuli are well-defined.

If γ_α is a boundary circle we set $C^-_n := C_n \backslash A^-_{n,\alpha,j}$. Otherwise γ_α separates two pants, say S_j and S_k. Then we define in a similar way the annuli $A_{n,\alpha,k} \subset \sigma_n(S_k)$ and $A^-_{n,\alpha,k} \subset \sigma_n(S_k)$, set $A_{n,\alpha} := A_{n,\alpha,j} \cup A_{n,\alpha,k}$ and $A^-_{n,\alpha} := A^-_{n,\alpha,j} \cup A^-_{n,\alpha,k}$, and put $C^-_n := C_n \backslash A^-_{n,\alpha}$.

The parametrizations $\sigma_n : \Sigma \to C_n$ can be chosen in such a way that the annuli $\sigma_n^{-1}(A^-_{n,\alpha,j})$ (resp. $\sigma_n^{-1}(A^-_{n,\alpha,k})$) define the same annulus $A^-_{\alpha,j}$ (resp. $A^-_{\alpha,k}$) on Σ. Let $\gamma^-_{\alpha,j}$ (resp. $\gamma^-_{\alpha,k}$) denote its boundary circles different from γ_α. Thus, the curves C^-_n are parametrized by a real surface $\Sigma^- := \Sigma \backslash A^-_{\alpha,j}$ (resp. $\Sigma^- := \Sigma \backslash (A^-_{\alpha,j} \cup A^-_{\alpha,k})$, and the restrictions of σ_n can be chosen as parametrization maps. Thus the decompositions of components of C_n into pants define the combinatorial type of decompositions of components of C^-_n into pants. Moreover, the corresponding graph Γ^- will be the same for all C^-_n. It coincides with Γ if γ_α is a boundary circle. Otherwise Γ^- can be obtained from Γ by replacing the edge corresponding to γ_α by 2 new tails for 2 new boundary components.

Let $C_{n,i}$ be the component of C_n adjacent to $\sigma_n(\gamma_\alpha)$ and $C^-_{n,i}$ the corresponding componet of C^-_n. Decompose $C^-_{n,i}$ into pants according to graph Γ^- in the canonical way, so that the boundary circles of the obtained pants are geodesic. Note that even if the constructed pants are in combinatorial one-to-one correspondence with the pants of C_n, the intrinsic metric on $C^-_{n,i}$ and the obtained geodesics circle γ^-_β differs from the corresponding objects on $C_{n,i}$

Nevertheless, we claim that for the obtained decomposition of $C^-_{n,i}$ the intrinsic lengths are uniformly bounded from above (possibly by a new constant) and that the sequence $\{C^-_n\}$ has less "shrinking circles" than $\{C_n\}$. This explains the meaning of the above construction, when the curves C^-_n are obtained from C^-_n by cutting off the annuli $A^-_{n,\alpha,j}$ (and resp. the annuli $A^-_{n,\alpha,j}$). The annuli we choose are sufficiently

long so that one "shrinking circle" disappears, but not too long so that the complex structures of the curves C^-_n remain non-degenerating near the boundary.

Indeed, the complex structures on C^-_n do not degenerate at the boundary circle $\gamma^-_{\alpha,j}$ (resp. at $\gamma^-_{\alpha,k}$), since C^-_n contain annuli $A_{n,\alpha,j}\backslash A^-_{n,\alpha,j}$ (resp. $A_{n,\alpha,k}\backslash A^-_{n,\alpha,k}$) of the constant conformal radius $R = e > 1$. This implies that the lengths $\ell^-_n(\gamma^-_{\alpha,j})$ of $\sigma_n(\gamma^-_{\alpha,j})$ (resp. $\ell^-_n(\gamma^-_{\alpha,k})$ of $\sigma_n(\gamma^-_{\alpha,k})$) with respect to the intrinsic metrics on C^-_n are uniformly bounded.

On the other hand, the lengths $\ell^-_n(\gamma^-_{\alpha,j})$ (resp. $\ell^-_n(\gamma^-_{\alpha,k})$) are also uniformly bounded from below by a positive constant. Otherwise, by *Lemma 3.4*, after passing to a subsequence, there would exist annuli $A_n \subset C^-_n$ of infinitely increasing radii R_n, adjacent to $\sigma_n(\gamma^-_{\alpha,j})$ (resp. to $\sigma_n(\gamma^-_{\alpha,k})$). The superadditivity of the logarithm of the conformal radius of annuli, see [Ab], Ch.II, § 1.3, shows that the conformal radius R^+_n of the annulus $A^-_{n,\alpha,j}\cup A_n$ satisfies the inequality $\log R^+_n \geqslant \frac{\pi^2}{\ell_{n,\alpha}} - \frac{2\pi}{a^*} - 1 + \log R_n$, which contradicts *Lemma 2.2*, part *i)*.

Now we estimate the intrinsic lengths of boundary circles and the number of "shrinking circles" on $C^-_{n,i}$. Compute the width L_n of $A_{n,\alpha,j}\backslash A^-_{n,\alpha,j}$ w.r.t. the intrinsic metric on C_n. Using $\ell_{n,\alpha} \longrightarrow 0$, we get

$$L_n = \int_{\rho=\frac{\pi^2}{\ell_{n,\alpha}}-\frac{2\pi}{a^*}-1}^{\frac{\pi^2}{\ell_{n,\alpha}}-\frac{2\pi}{a^*}} \left(\frac{\frac{\ell_{n,\alpha}}{2\pi}}{\cos\frac{\ell_{n,\alpha}\rho}{2\pi}}\right) d\rho = \left[\log\operatorname{cotan}\left(\frac{\pi}{4} - \frac{\ell_{n,\alpha}\rho}{4\pi}\right)\right]_{\rho=\frac{\pi^2}{\ell_{n,\alpha}}-\frac{2\pi}{a^*}-1}^{\frac{\pi^2}{\ell_{n,\alpha}}-\frac{2\pi}{a^*}}$$

$$= \log\frac{\operatorname{cotan}\frac{\ell_{n,\alpha}}{2a^*}}{\operatorname{cotan}(\frac{\ell_{n,\alpha}}{2a^*}+\frac{\ell_{n,\alpha}}{4\pi})} \approx \log\frac{\frac{\ell_{n,\alpha}}{2a^*}+\frac{\ell_{n,\alpha}}{4\pi}}{\frac{\ell_{n,\alpha}}{2a^*}} = \log\left(1+\frac{a^*}{2\pi}\right) > 0.$$

Let $\gamma_\beta \neq \gamma_\alpha$ be another boundary circle of Σ, such that the circles $\sigma_n(\gamma_\beta)$ bound $C_{n,i}$. Denote by $\ell_{n,\beta}$ (resp. by $\ell^-_{n,\beta}$) the length of $\sigma_n(\gamma_\beta)$ w.r.t. the intrinsic length of $C_{n,i}$ (resp. of $C^-_{n,i}$). By *Lemma 2.2* for any n we can find an annulus $A_{n,\beta} \subset C_n$ of the constant width which is adjacent to $\sigma_n(\gamma_\beta)$ and has the conformal radius $R_{n,\beta}$ with $\log R_{n,\beta} = \frac{\pi^2}{\ell_{n,\beta}} - \frac{2\pi}{a^*}$. The width of $A_{n,\beta}$ is then

$$L_{n,\beta} = \log\operatorname{cotan}\left(\frac{\pi}{4} - \frac{\ell_{n,\beta}}{4\pi}\left(\frac{\pi^2}{\ell_{n,\beta}} - \frac{2\pi}{a^*}\right)\right) = \log\operatorname{cotan}\left(\frac{\ell_{n,\beta}}{2a^*}\right).$$

Since $\ell_{n,\beta}$ are uniformly bounded from above, $L_{n,\beta}$ are uniformly bounded from below. This implies that we can find a subannuli $A^-_{n,\beta} \subset C^-_{n,i}$, which are ajacent to $\sigma_n(\gamma_\beta)$ and have the constant width $L^* > 0$ w.r.t. the intrinsic metric of $C_{n,i}$. Computing the conformal radius $R_{n,\beta}$ of such an annulus $A^-_{n,\beta}$ we get the relation

$$L^* = \log\operatorname{cotan}\left(\frac{\pi}{4} - \frac{R_{n,\beta}\ell_{n,\beta}}{4\pi}\right).$$

Conseqently, $R_{n,\beta} = \frac{M}{\ell_{n,\beta}}$ for some constant $M > 0$. Since $\ell_{n,\beta}$ are uniformly bounded from above, $R_{n,\beta}$ are uniformly bounded from below. This means that the complex structures of C^-_n do not degenerate near boundary.

Now we estimate the number of "shrinking circles" on C^-_n. Take a circle $\gamma \subset \Sigma^-$ such that for every n there exists a simple closed geodesic $\gamma^-_n \subset C^-_n$ homotopic to

$\sigma_n^-(\gamma)$. Assume that the intrinsic lengths $\ell^-(\gamma_n^-)$ vanish. Using *Lemma 2.2*, we can construct annuli $A_n \subset C_n^-$ homotopic to γ_n^- whose conformal radii R_n increase infinitely. But then there exist simple closed geodesics $\gamma_n \subset C_n$ homotopic to $\sigma_n(\gamma)$, and the $A_n \subset C_n$ are homotopic to γ_n. *Lemma 2.2* implies that the intrinsic lengths $\ell(\gamma_n)$ also vanish. Thus every "shrinking circle" on C_n^- appears from some "shrinking circle" on C_n.

Thus we have shown, that $\{C_n^-\}$ with the intrinsic metric and the defined by decomposition Γ^- have uniformly bounded lengths of marked circles and less "shrinking circles" than $\{C_n\}$. By induction, we may assume that there exist the covering of Σ^- and parametrizations of C_n^- by Σ^- with the desired properties. Since $C_n \backslash C_n^-$ are annuli of increasing conformal radii, the statement of the theorem is valid for C_n. □

Lemma 4.3. *Let C_n be a sequence of annuli with complex structures j_n, Σ some fixed annulus, and $\delta_n : \Sigma \to C_n$ some parametrizations. Suppose that for two fixed annuli $A_1, A_2 \subset \Sigma$ adjacent to the boundary circles of Σ the restrictions $\delta_n^* j_n|_{A_i}$ do not depend on n.*

Then one can find parametrizations $\sigma_n : \Sigma \to C_n$ such that σ_n coincide with δ_n on some (possibly smaller) annuli A_i', also adjacent to the boundary circles of Σ restrictions, and such that:

i) if conformal radii R_n of C_n converge to $R_\infty < \infty$, then $\sigma_n^* j_n$ converge to some complex structure;

ii) if conformal radii R_n of C_n converge to ∞, then for some circle $\gamma \subset \Sigma$ structures $\sigma_n^* j_n$ converge on compact subsets $K \Subset \Sigma \backslash \gamma$ to the complex structure of the disjoint union of two punctured disks. Moreover, as such γ an arbitrary imbedded circle generating $\pi_1(\Sigma)$ can be chosen.

Proof. Without loss of generality we may assume that $\Sigma = A(1,10)$, $A_1 = A(7,10)$, $A_2 = A(1,4)$ and that the given circle γ lies in $A(3,7)$. Let $\delta_n : \Sigma \to C_n$ be the given parametrizations. There exist biholomorphisms $\varphi_n : C_n \to A(r_n, 1)$ with $r_n^{-1} = R_n$ being the conformal radii of C_n, such that $\varphi_n(\delta_n(A_1))$ is adjacent to $\{|z| = 1\} = \partial\Delta$ and $\varphi_n(\delta_n(A_2))$ is adjacent to $\{|z| = r_n\}$. Define $\varphi_n'(z) := \frac{r_n}{\varphi_n(z)}$.

Recall that structures $\delta_n^* j_n|_{A_i}$ are the same for all n. We call this structure j. Consider the maps $\varphi_n \circ \delta_n : (A_1, j) \to A(r_n, 1) \subset \Delta$ and $\varphi_n' \circ \delta_n : (A_2, j) \to A(r_n, 1) \subset \Delta$. Passing to a subsequence we can suppose that $r_n \longrightarrow r_\infty < 1$ and that the maps $\varphi_n \circ \delta_n$, $\varphi_n' \circ \delta_n$ converge on $A_1 \cup A_2$ to holomorphic maps $\psi : (A_1 \cup A_2, j) \to \Delta$ and $\psi' : (A_1 \cup A_2, j) \to \Delta$ respectively. This means that the maps $(\varphi_n \circ \delta_n, \varphi_n' \circ \delta_n) : (A_1 \cup A_2) \to \Delta^2$ take values in $\{(z,z') \in \Delta^2 \,:\, z{\cdot}z' = r_n\}$ and converge to the map $(\psi, \psi') : (A_1 \cup A_2) \to \Delta^2$ with values in $\{(z,z') \in \Delta^2 \,:\, z{\cdot}z' = r_\infty\}$.

The arguments from the proof of *Lemma 4.1* show that the annuli $\psi(A_1)$ and $\psi'(A_2)$ are adjacent to $\partial\Delta$. This implies that for $n >> 1$ there exist diffeomorphisms $(\psi_n, \psi_n') : \Sigma \to \{(z,z') \in \Delta^2 \,:\, z{\cdot}z' = r_n\}$ such that $\psi_n \equiv \varphi_n \circ \delta_n$ on $A(9,10)$, $\psi_n'(z) \equiv \varphi_n' \circ \delta_n(z)$ for $z \in A(1,2)$, and (ψ_n, ψ_n') converge to (ψ, ψ') on Σ. Moreover, we may assume that $|\psi_n(t)| = |\psi_n'(t)| = \sqrt{r_n}$ for any $t \in \gamma$. This means that $(\psi_n, \psi_n')(\gamma)$ lies on the middle circle $\{(z,z') : |z| = \sqrt{r_n},\ z' = \frac{r_n}{z}\}$ of $\{(z,z') \in \Delta^2 \,:\, z{\cdot}z' = r_n\}$.

Set $\sigma_n := \varphi_n^{-1} \circ \psi_n : \Sigma \to C_n$. Then, obviously, $\sigma_n \equiv \delta_n$ on $A(1,2)$ and on $A(9,10)$, and $\sigma_n^* j_n = (\psi_n, \psi')_n^* J_{\mathrm{st}} \longrightarrow (\psi, \psi')^* J_{\mathrm{st}}$, where J_{st} denotes the standard complex structure on Δ^2. □

Proof of Theorem 1.1. Let $\{(C_n, u_n)\}$ be the sequence from the hypothesis of the theorem. Then the condition *c)* of the theorem and *Lemma 4.1* yields the existence of parametrizations $\delta_n : \Sigma \to C_n$ and annuli A_i, adjacent to each boundary circle γ_i, such that $\delta_n^* j_{C_n}$ are constant in every A_i. Thus we may assume that δ_n with these properties are given.

Take a covering $\mathcal{V} = \{V_\alpha\}$ and parametrizations σ_n as in *Theorem 4.3*. With every such covering we can associate the curves $C_{\alpha,n} := \sigma_n(V_\alpha)$, the parametrizations $\sigma_{\alpha,n} := \sigma_n\big|_{V_\alpha} : V_\alpha \to C_{\alpha,n}$, and the maps $u_{\alpha,n} := u_n\big|_{C_{\alpha,n}} : C_{\alpha,n} \to X$. Consider the following type of convergence of sequences $\{(C_{\alpha,n}, u_{\alpha,n}, \sigma_{\alpha,n})\}$ with α fixed:

A) *$C_{\alpha,n}$ are annuli of infinitely growing conformal radii l_n and the conclusions of Lemma 3.7 hold;*

B) *every $C_{\alpha,n}$ is isomorphic to the standard node $\mathcal{A}_0 = \Delta \cup_{\{0\}} \Delta$, such that the compositions $V_\alpha \xrightarrow{\sigma_{\alpha,n}} C_{\alpha,n} \xrightarrow{\cong} \mathcal{A}_0$ define the same parametrizations of $\mathcal{A}_0$ for all n; furthermore, the induced maps $\tilde{u}_{\alpha,n} : \mathcal{A}_0 \to X$ strongly converge;*

C) *the structures $\sigma_n^* j_n\big|_{V_\alpha}$ and the maps $u_{\alpha,n} \circ \sigma_{\alpha,n} : V_\alpha \to X$ strongly converge.*

Here the strong convergence of maps is the one in the $L^{1,p}$-topology on compact subsets for some $p > 2$ (and hence for all $p < \infty$), and the convergence of structures means the usual C^∞-convergence.

Suppose that there is a subsequence, still indexed by $n \to \infty$, such that for any V_α we have one of the convergence types *A)–C)*. Then the sequence of global maps $\{(C_n, u_n, \sigma_n)\}$ converges in the Gromov topology which gives us the proof, and also a precise description of the convergence.

Otherwise, we want to find a refinement of our covering $\mathcal{V}$ and parametrizations σ_n which have the needed properties. We shall proceed by induction estimating the area of pieces of coverings of Σ. To do so, we fix $\varepsilon > 0$ satisfying $\varepsilon \leqslant \frac{\varepsilon_1^2}{2}$ with ε_1 from *Lemma 3.1*, $\varepsilon \leqslant \frac{\varepsilon_2^2}{2}$ with ε_2 from *Lemma 3.3*, and $\varepsilon \leqslant \frac{\varepsilon_3}{3}$ with ε_3 from *Lemma 3.8*. Consider first

Special case: $\mathsf{area}\,(u_n(\sigma_n(V_\alpha))) \leqslant \varepsilon$ *for any n and any $V_\alpha \in \mathcal{V}$.* We can consider every V_α separately. If the structures $\sigma_n^* j_n|_{V_\alpha}$ are constant, then, due to *Corollary 3.3*, some subsequence of $u_n \circ \sigma_n$ strongly converges.

If structures $\sigma_n^* j_n|_{V_\alpha}$ are not constant, then V_α must be an annulus. Fix biholomorphisms $\varphi_n : Z(0, l_n) \xrightarrow{\cong} \sigma_n(V_\alpha)$. If $l_n \longrightarrow \infty$, then *Lemma 3.7* shows that (and describes how!) an appropriate subsequence of $u_n \circ \varphi_n$ converges to a J_∞-holomorphic map of a standard node. Otherwise we can find a subsequence, still denoted (C_n, u_n), for which $l_n \longrightarrow l_\infty < \infty$ and $u_n \circ \varphi_n$ converge to a J_∞-holomorphic map of $Z(0, l_\infty)$ in $L^{1,p}$-topology on compact subsets $K \Subset Z(0, l_\infty)$ for any $p < \infty$. To construct refined parametrizations $\tilde{\sigma}_{\alpha,n} : V_\alpha \to C_{\alpha,n} = \sigma_n(V_\alpha)$, we use property *(d)* of *Theorem 4.2* and apply *Lemma 4.3*.

Thus we get one of the convergence types *A)–C)* which completes the proof in *Special case*.

General case. Suppose that the theorem is proved for all sequences of J_n-holomorphic curves $\{(C_n, u_n)\}$ with parametrizations $\delta_n : \Sigma \to C_n$ which satisfy the additional condition $\mathsf{area}\,(u_n(C_n)) \leqslant (N-1)\varepsilon$ for all n. We consider this as the induction hypothesis in N, so that our *Special case* is the base of the induction.

Assume that there exists a subsequence, still indexed by $n \to \infty$, such that for every V_α and for the curves $C_{\alpha,n} = \sigma_n(V_\alpha)$ the statement of the theorem holds. This means the existence of refined coverings $V_\alpha = \cup_i V_{\alpha,i}$ and new parametrizations $\tilde\sigma_{\alpha,n} : V_\alpha \to C_{\alpha,n}$, such that $\tilde\sigma_n$ coincide with σ_n near the boundary of every V_α and such that for curves $C_{\alpha,i,n} := \tilde\sigma_n(V_{\alpha,i})$ we have the convergence of one of the types *A)–C)*. Then we can glue $\tilde\sigma_{\alpha,n}$ together to global parametrizations $\tilde\sigma_n : \Sigma \to C_n$ and set $\widetilde{\mathcal{V}} := \{V_{\alpha,i}\}$, getting the proof.

In particular, it is so for any V_α, such that $\mathsf{area}\,(u_n(\sigma_n(V_\alpha))) \leqslant (N-1)\varepsilon$ for all n due to inductive hypothesis.

This implies that it is sufficient to consider only those V_α, for which $(N-1)\varepsilon \leqslant \mathsf{area}\,(u_n(\sigma_n(V_\alpha))) \leqslant N\varepsilon$ for all n. Obviously, it is sufficient to show the desired property only for one such piece of covering, say for V_1. To construct the refined parametrizations $\tilde\sigma_{1,n}$ and the covering $V_1 = \cup_i V_{1,i}$, we consider 4 cases:

Case 1): The structures $\sigma_n^* j_n|_{V_1}$ do not change and $C_{1,n}$ are not isomorphic to the standard node $\mathcal{A}_0$. Then we can realize $(V_1, \sigma_n^* j_n)$ as a constant bounded domain D in $\mathbb{C}$. Hence we can consider $u_n \circ \sigma_n : V_\alpha \to X$ as pseudoholomorphic maps $u_n : D \to (X, J_n)$. Now we use the "patching construction" of Sacks-Uhlenbeck [S-U].

Fix some $a > 0$. Denote $D_{-a} := \{z \in D \ : \ \Delta(z,a) \subset D\}$. Find a covering of D_{-a} by open sets $U_i \subset D$ with $\mathsf{diam}(U_i) < a$, such that any $z \in D$ lies in at most 3 pieces U_i. Then for any n there exists at most $3N$ pieces U_i with $\mathsf{area}\,(u_n(U_i)) > \varepsilon$. Taking a subsequence, we may assume that the set of such "bad" pieces U_i is the same for all n. Repeat successively the same procedure for $\frac{a}{2}$, $\frac{a}{4}$, and so on, and then take the diagonal subsequence. We obtain at most $3N$ "bad" points $y_1^*, \dots, y_l^*$, such that a subsequence of u_n converges in $D \backslash \{y_1^*, \dots, y_l^*\}$ strongly, *i.e.* in the $L^{1,p}$-topology on compact subsets $K \Subset D \backslash \{y_1^*, \dots, y_l^*\}$. These "bad" points $y_1^*, \dots, y_l^*$ are characterized by the following property:

$$\text{for any } r > 0 \ \ \mathsf{area}\,(u_n(\Delta(y_i^*, r)) > \varepsilon \ \text{ for } n \text{ all sufficiently big.} \tag{4.1}$$

Remark. As we shall see now, every such point is a place where the "bubbling" occurs. Therefore we shall call y_i^* *bubbling points*. The characterization property of a bubbling point is (4.1).

If there are no bubbling points, *i.e.* $l = 0$, then the chosen subsequence u_n converges strongly and we can finish the proof by induction.

Otherwise we consider the first point $y_1^* \in D$. Take a disk $\Delta(y_1^*, \varrho)$ which doesn't contain any other bubbling points y_i^*, $i > 1$.

Then for any n we can find the unique r_n such that

(1) $r_n \leqslant \frac{\varrho}{2}$ and $\mathsf{area}\,(u_n(\Delta(x, r_n))) \leqslant \varepsilon$ for any $x \in \overline{\Delta}(y_1^*, \frac{\varrho}{2})$;

(2) r_n is maximal w.r.t. (1).

Then $r_n \longrightarrow 0$, since otherwise for $r^+ := \mathsf{lim\,sup}\, r_n > 0$ and for some subsequence $n_k \longrightarrow \infty$ with $r_{n_k} \longrightarrow r^+$ we would have

$$\mathsf{area}\left(u_{n_k}(\Delta(y_1^*, r^+))\right) \leqslant \varepsilon,$$

contradicting to (4.1).

Lemma 4.4. *For every $n >> 1$ there exists $x_n \in \overline{\Delta}(y_1^*, \frac{\varrho}{2})$, such that $x_n \to y_1^*$ and* $\mathsf{area}\,(u_n(\Delta(x_n, r_n))) = \varepsilon$.

Proof. If not, then for some subsequence $n_k \longrightarrow \infty$ and every $x \in \overline{\Delta}(y_1^*, \frac{\varrho}{2})$ we would have $\mathsf{area}\,(u_{n_k}(\Delta(x, r_{n_k}))) < \varepsilon$. Since $r_n \longrightarrow 0$, this would contradict with the maximality of r_n. In particular, there exists a sequence $\{x_n\}$ with the desired properties.

If x_n do not converge to y_1^*, then after going to a subsequence we would find $y' = \mathsf{lim}_{n\to\infty} x_n \neq y_1^*$. By our construction, y' does not coincide with any other bubbling point y_i^*. Take $a > 0$ such that $\Delta(y', a)$ contains no bubbling point. Then by *Corollary 3.3* some subsequence u_{n_k} would converge to some $u' \in L^{1,p}_{\text{loc}}(\Delta(y', a), X)$ in strong $L^{1,p}(K)$-topology for any compact subset $K \Subset \Delta(y', a)$ and any $p < \infty$. In particular, for sufficiently small $b < a$ we would have $\mathsf{area}\,(u_{n_k}(\Delta(y', b))) \longrightarrow \mathsf{area}\,(u'(\Delta(y', b))) < \varepsilon$, which would contradict the choice of r_n and x_n. $\square$

Using r_n and x_n constructed above, define the maps $v_n : \Delta(0, \frac{\varrho}{2r_n}) \to (X, J_n)$ by $v_n(z) := u_n(x_n + r_n z)$. By the definition of r_n we have

$$\mathsf{area}\,(v_n(\Delta(x,1)) \leqslant \varepsilon \quad \text{for all } x \in \Delta(0, \tfrac{\varrho}{2r_n} - 1). \tag{4.2}$$

On the other hand, $\mathsf{area}\,(v_n(\Delta(0,1)) = \mathsf{area}\,(u_n(\Delta(x_n, r_n)) = \varepsilon$ by *Lemma 4.4*. Thus v_n converge (after going to a subsequence) on compact subsets in $\mathbb{C}$ to a nonconstant J_∞-holomorphic map v_∞ with finite energy. Consequently, v_∞ extends to S^2 by the removable singularity theorem of *Corollary 3.6*.

Since v_∞ is nonconstant, $\|dv_\infty\|^2_{L^2(S^2)} = \mathsf{area}\,(v_\infty(S^2)) \geqslant 3\varepsilon$ by *Lemma 3.8* and the choice of ε. Choose $b > 0$ in such a way that

$$\mathsf{area}\,(v_\infty(\Delta(0,b)) = \|dv_\infty\|^2_{L^2(\Delta(0,b))} \geqslant 2\varepsilon. \tag{4.3}$$

By *Corollary 3.3* this implies that

$$\|du_n\|^2_{L^2(\Delta(x_n, br_n))} = \|dv_n\|^2_{L^2(\Delta(0,b))} \geqslant \varepsilon. \tag{4.4}$$

For $n >>$ we consider the coverings of V_1 by 3 sets

$$V^{(n)}_{1,1} := V_1 \backslash \overline{\Delta}(y_1^*, \tfrac{\varrho}{2}), \qquad V^{(n)}_{1,2} := \Delta(y_1^*, \varrho) \backslash \overline{\Delta}(x_n, br_n), \qquad V^{(n)}_{1,3} := \Delta(x_n, 2br_n).$$

Fix n_0 sufficiently big. Denote $V_{1,1} := V^{(n_0)}_{1,1}$, $V_{1,2} := V^{(n_0)}_{1,2}$, and $V_{1,3} := V^{(n_0)}_{1,3}$. There exist diffeomorphisms $\psi_n : V_1 \to V_1$ such that $\psi_n : V_{1,1} \to V^{(n)}_{1,1}$ is the identity, $\psi_n : V_{1,2} \to V^{(n)}_{1,2}$ is a diffeomorphism, and $\psi_n : V_{1,3} \to V^{(n)}_{1,3}$ is biholomorphic w.r.t. the complex structures, induced from $C_{1,n}$.

Thus we have constructed the covering $\{V_{1,1}, V_{1,2}, V_{1,3}\}$ of V_1 and parametrizations $\sigma'_n := \sigma_{1,n} \circ \psi_n : V_1 \to C_{1,n}$, such that the conditions of *Theorem 4.2* are satisfied. Moreover, $\mathsf{area}\,(u_n(\sigma'_n(V_{1,i}))) \leqslant (N-1)\varepsilon$ due to inequality (4.4). Consequently, we can apply the inductive assumptions to the sequence of curves $\sigma'_n(V_{1,i})$ and finish the proof by induction.

Case 2): V_1 is a cylinder, structures $\sigma_n^* j_n|_{V_1}$ vary with n, but conformal radii of $(V_1, \sigma_n^* j_n)$ are bounded uniformly in n. Applying *Lemma 4.3* we can assume

that structures $\sigma_n^* j_n$ converge to a structure of an annulus with finite conformal radius. The constructions of *Case 1)* go through here with the following minor modifications. Firstly, we find the set of the bubbling points $y_i^* \in V_1$, using the same patching construction and the characterization (4.1). Then we find diffeomorphisms $\varphi_n : V_1 \to V_1$, such that *a)* φ_n converge to the identity map $\mathsf{Id} : V_1 \to V_1$; *b)* φ_n are identical in fixed (*i.e.* independent of n) annuli adjacent to the boundary circles of V_1; *c)* φ_n preserve every bubbling point, $\varphi_n(y_i^*) = y_i^*$; and finally *d)* for the "corrected" parametrizations $\tilde\sigma_n := \sigma_n \circ \varphi_n$ the structures $\tilde\sigma_n^* j_n|_{V_1}$ are constant in a neighborhood of every bubbling point y_i^*. Then we repeat remaining constructions of *Case 1)* using the new parametrizations $\tilde\sigma_n$.

Case 3): Every $C_{1,n} = \sigma_n(V_1)$ is isomorphic to the standard node $\mathcal{A}_0$. Fix identifications $C_{1,n} \cong \mathcal{A}_0$ such that the induced parametrization maps $\sigma_{1,n} : V_1 \to \mathcal{A}_0$ are the same for all n. Represent $\mathcal{A}_0$, and hence every $C_{1,n}$, as the union of two discs Δ' and Δ'' with identification of the centers $0 \in \Delta'$ and $0 \in \Delta''$ into the nodal point of $\mathcal{A}_0$, still denoted by 0. Let $u_n' : \Delta' \to X$ and $u_n'' : \Delta'' \to X$ be the corresponding "components" of the maps $u_{1,n} : C_{1,n} \to X$. Find the common collection of bubbling points y_i^* for both maps $u_n' : \Delta' \to X$ and $u_n'' : \Delta'' \to X$. If there are no bubbling points, then we obtain the convergence type *B)* and the proof can be finished by induction. Otherwise, we consider the first such point y_1^*, which lies, say, on Δ'. If y_1^* is distinct from the nodal point $0 \in \Delta'$, then we simply repeat all the constructions of *Case 1)*.

It remains to consider the case $y_1^* = 0 \in \Delta'$. Now one should modify the arguments of *Case 1* in the following way. Construct the sequences of radii $r_n \longrightarrow 0$, of points $x_n \longrightarrow y_1^* = 0$, and of maps $v_n : \Delta(0, \frac{\varrho}{2r_n}) \to X$, $v_\infty : S^2 \to X$ as in *Case 1*. Set $R_n := |x_n|$, so that R_n is the distance from x_n to point $0 = y_1^* \in \Delta'$. After rescaling u_n to the maps v_n, the point $0 \in \Delta'$ will correspond to the point $z_n^* := -\frac{x_n}{r_n}$ in the definition domain $\Delta(0, \frac{\varrho}{2r_n})$ of the map v_n. We consider 2 subcases.

Subcase 3′): The sequence $\frac{R_n}{r_n}$ is bounded. This is equivalent to boundednes of the sequence z_n^*. Going to a subsequence we may assume that the sequence z_n^* converges to a point $z^* \in \mathbb{C}$. This point will be a nodal one for (S^2, v_∞). As above, v_∞ is nonconstant and $\|dv_\infty\|^2_{L^2(S^2)} = \mathsf{area}\,(v_\infty(S^2)) \geqslant 3\varepsilon$. Choose $b > 0$ in such a way that

$$\|dv_\infty\|^2_{L^2(\Delta(0,b))} \geqslant 2\varepsilon \tag{4.5}$$

and $b \geqslant 2|z^*| + 2$. Due to *Corollary 3.3* for $n >> 1$ we obtain the estimate

$$\|du_n'\|^2_{L^2(\Delta'(x_n, br_n))} = \|dv_n\|^2_{L^2(\Delta(0,b))} \geqslant \varepsilon. \tag{4.6}$$

Here $\Delta'(x,r)$ denotes the subdisc of Δ' with center x and radius r. Furthemore, for $n >> 1$ we have the relation $z_n^* \in \Delta(0, b-1)$, or equivalently, $0 \in \Delta'(x_n, (b-1)r_n))$.

Define the coverings of $\mathcal{A}_0$ by 4 sets

$$W_1^{(n)} := \Delta' \backslash \overline{\Delta}'(0, \tfrac{\varrho}{2}), \qquad W_2^{(n)} := \Delta'(0,\varrho) \backslash \overline{\Delta}'(x_n, br_n),$$
$$W_3^{(n)} := \Delta'(x_n, 2br_n) \backslash \overline{\Delta}'(0, \tfrac{r_n}{2}), \quad W_4^{(n)} := \Delta'(0, r_n) \cup \Delta'',$$

and lift them to V_1 by putting $V_{1,i}^{(n)} := \sigma_{1,n}^{-1}(W_i^{(n)})$. Choose $n_0 >> 0$, such that $z_{n_0}^* \in \Delta(0, b-1)$ and the relation (4.6) holds. Set $V_{1,i} := V_{1,i}^{(n_0)}$. Choose diffeomorphisms

$\psi_n : V_1 \to V_1$ such that $\psi_n : V_{1,1} \to V_{1,1}^{(n)}$ is the identity map, $\psi_n : V_{1,2} \to V_{1,2}^{(n)}$ and $\psi_n : V_{1,3} \to V_{1,3}^{(n)}$ are diffeomorphisms, and $\psi_n : V_{1,4} \to V_{1,4}^{(n)}$ is corresponds to isomorphisms of nodes $W_4^{(n)} \cong \mathcal{A}_0$. Set $\sigma_n' := \sigma_n \circ \psi_n$. The choice above can be done in such a way that the refined covering $\{V_{1,i}\}$ of V_1 and parametrization maps $\sigma_n' : V_1 \to C_{1,n}$ have the properties of *Theorem 4.2.* Moreover, relations (4.2) and (4.6) imply the estimate $\mathsf{area}\,(u_n(\sigma_n'(V_{1,i})) \leqslant (N-1)\varepsilon$. This yields the inductive conclusion in *Subcase 3′).*

Subcase 3″): The sequence $\frac{R_n}{r_n}$ increase infinitely. This means that the sequencez_n^* is not bounded. Nevertheless $R_n \longrightarrow 0$ since $x_n \longrightarrow 0$. We proceed as follows. Construct of the radius b as in *Case 1).* For $n >> 0$ define the coverings of $\mathcal{A}_0$ by 6 sets

$$
\begin{aligned}
&W_1^{(n)} := \Delta' \backslash \overline{\Delta}'(0, \tfrac{\varrho}{2}), && W_2^{(n)} := \Delta'(0,\varrho) \backslash \overline{\Delta}'(x_n, 2R_n),\\
&W_3^{(n)} := \Delta'(x_n, 4R_n) \backslash \big(\overline{\Delta}'(x_n, \tfrac{R_n}{6}) \cup \overline{\Delta}'(0, \tfrac{R_n}{6})\big) && W_4^{(n)} := \Delta'(0, \tfrac{R_n}{3}) \cup \Delta'',\\
&W_5^{(n)} := \Delta'(x_n, \tfrac{R_n}{3}) \backslash \overline{\Delta}'(x_n, br_n), && W_6^{(n)} := \Delta'(0, 2br_n),
\end{aligned}
$$

and lift them to V_1 by putting $V_{1,i}^{(n)} := \sigma_{1,n}^{-1}(W_i^{(n)})$. Choose $n_0 >> 0$, such that $R_{n_0} >> br_{n_0}$, and set $V_{1,i} := V_{1,i}^{(n_0)}$. Choose diffeomorphisms $\psi_n : V_1 \to V_1$ such that $\psi_n : V_{1,1} \to V_{1,1}^{(n)}$ is the identity map, $\psi_n : V_{1,2} \to V_{1,2}^{(n)}$, $\psi_n : V_{1,4} \to V_{1,4}^{(n)}$ and $\psi_n : V_{1,5} \to V_{1,5}^{(n)}$ are diffeomorphisms, and finally, $\psi_n : V_{1,6} \to V_{1,6}^{(n)}$ corresponds to isomorphisms of nodes $W_6^{(n)} \cong \mathcal{A}_0$. Set $\sigma_n' := \sigma_n \circ \psi_n$. Again, this choice can be done in such a way that $\{V_{1,i}\}$ and parametrization maps $\sigma_n' : V_1 \to C_{1,n}$ have the properties of *Theorem 4.2.* As above, we get the estimate $\mathsf{area}\,(u_n(\sigma_n'(V_{1,i})) \leqslant (N-1)\varepsilon$ due to (4.2). Thus we get the inductive conclusion for *Subcase 3″)* and can proceed further.

Case 4): V_1 is a cylinder, structures $\sigma_n^* j_n|_{V_1}$ vary with n, and conformal radii of $(V_1, \sigma_n^* j_n)$ converge to $+\infty$. Using *Lemma 4.3* we can assume that structures $\sigma_n^* j_n$ satisfy property *ii)* of this lemma.

Fix biholomorphisms $\sigma_n(V_1) \cong Z(0, l_n)$. If $\mathsf{area}\,(u_n(Z(a-1,a))) \leqslant \varepsilon$ for any n and any $a \in [1, l_n]$, then *Lemma 3.7* shows that $u_n : \sigma_n(V_1) \to X$ converge to a J_∞-holomorphic map from node.

If not, then, after passing to a subsequence, we can find a sequence $\{a_n\}$ with $a_n \in [1, l_n]$, such that $\mathsf{area}\,(u_n(Z(a_n - 1, a_n))) \geqslant \varepsilon$. If a_n is bounded, say $a_n \leqslant a^+$, then we cover $Z(0, l_n)$ by the sets $V_{1,1} := Z(0, a^+ + 2)$ and $V_{1,2} := Z(a^+ + 1, l_n)$. If $l_n - a_n$ is bounded, say $l_n - a_n \leqslant a^+$, then we cover $Z(0, l_n)$ by the sets $V_{1,1} := Z(0, l_n - a^+ + 2)$ and $V_{1,2} := Z(l_n - a^+ + 1, l_n)$. In the remaning case, when both a_n and $l_n - a_n$ increase infinitely, we cover $Z(0, l_n)$ by 3 sets $V_{1,1}^{(n)} := Z(0, a_n - 1)$, $V_{1,2}^{(n)} := Z(a_n - 2, a_n + 1)$, and $V_{1,3}^{(n)} := Z(a_n, l_n)$.

Identify V_1 with the cylinder $Z(0,5)$ and cover it by 2 or respectively 3 successive cylinders $V_{1,i}$, *e.g.* by 2 cylinders $V_{1,1} := Z(0,2)$ and $V_{1,2} := Z(1,5)$, or respectively by 3 cylinders $V_{1,1} := Z(0,2)$, $V_{1,2} := Z(1,4)$, and $V_{1,3} := Z(3,5)$. Find diffeomorphisms $\psi_n : V_1 \to V_1$, identical in the neighborhood of the boundary of V_1 and such that $\psi_n(V_{1,i}) = \sigma_n^{-1} V_{1,i}^{(n)}$. Define the new parametrizations $\sigma_n' := \sigma_n \circ \psi_n$. Note that

we may additionally assume that if the conformal radius of $\sigma'_n(V_{1,i})$ is independent of n then the structure $\sigma'^*_n j_n\big|_{V_{1,i}}$ is also is independent of n.

By the construction, we get the following property of the covering $\{V_{1,i}\}$ and new parametrizations σ'_n: For any V_i either we have the estimate

$$\mathsf{area}\,(u_n(\sigma'_n(V_{1,i}))) \leqslant (N-1)\,\varepsilon.$$

or the structures $\sigma'^*_n j_n|_{V_{1,i}}$ do not depend on n. Thus we reduce our case to the situation which is covered either by the inductive assumption or by *Case 1)*.

The proof of the theorem can be finished by induction. The fact that the limit curve (C_∞, u_∞) remains stable over X is proved in *Lemma 4.5* below. $\square$

Remark. We explain here the meaning of the constructions used in the proof. We start with *Case 1)* where J_n-holomorphic maps from a fixed domain $D \subset \mathbb{C}$ are treated. The bubbling points y_i^* appear in this case as those ones where the strong convergence of maps $u_n : D \to (X, J_n)$ fails. The patching construction of Sacks and Uhlenbeck insures us that the "convergence failure" set is finite and gives an effective estimate of the number of bubbling points by the upper bound of the area, $l \leqslant 3N$ in our situation. The characterization property (4.2) of bubbling points is essentially due to Sacks and Uhlenbeck; the only difference is that we use the area of the map u, *i.e.* L^2-norm of du, see *Definition 1.4* , whereas in [S-U] the $\|du\|_{L^\infty}$ is used. The next step, namely the construction of maps v_n as rescaling of the u_n and the existence of the limit v_∞, is also due to Sacks and Uhlenbeck.

The explicit construction of the map v_∞ suggests the interpretation of the curve (S^2, v_∞) as a "bubbled sphere" and y_1^* as the point where the "bubbling" occurs. Moreover, one obtains natural partitions (one for each $n >> 0$) of D into three pieces: D minus fixed small neighborhood of y_1^*; disks $(\Delta(x_n, br_n), u_n)$ representing pieces $(\Delta(0,b), v_\infty)$ and approximating a sufficiently big part $(\Delta(0,b), v_\infty)$ of the bubbled sphere; and the "part inbetween".

These latter "parts inbetween" appear to be the annuli of infinitely growing conformal radii, considered in *Case 4)*. Since neither outer nor inner boundary circle should be preferred in some way, we consider them as long cylinders $C_n = Z(0, l_n)$ with $l_n \longrightarrow \infty$ according to *Definition 3.2. Lemma 3.7* provides a "good" convergence model for long cylinders, stated above as convergence type *A)*. If for a sequence (C_n, u_n) such convergence fails, then there must exist subannuli $A_n \subset C_n$ of a constant conformal radius, for which $\mathsf{area}\,(u_n(A_n)) \geqslant \varepsilon$.

In both cases — a constant domain D or long cylinders — we proceed by cutting the curves into smaller pieces. The situation we come to is simpler in the following sense. The obtained curves either converge or have the upper bound for the area smaller by the fixed constant ε. Thus the induction leads finally to decomposition of the curves into pieces for which one of the convergence types *A)–C)* holds. The possibility to glue these final pieces together is insured by the fact that the partitions above are represented by appropriate coverings satisfying the conditions of *Theorem 4.2.*

Considering curves with nodes, an additional attention should be payed to the case when bubbling appears at a nodal point. This situation is considered in *Case 3)*. The constructed points x_n and radii r_n describe the "center" and the "size"

of energy localization of the bubbling, represented by the sequence of the maps v_n tending to v_∞. So the convergence picture depends on whether the energy localization occurs near the nodal point (*Subcase 3′)*) or away of it (*Subcase 3″)*). As a result, the nodal point can either remain on the "bubbled" sphere (S^2, v_∞) or move into the "part inbetween", which is represented by long cylinders.

In *Subcase 3′)* we remove neighbourhoods of the nodal point from the disks $(\Delta(0,b), v_n)$ and thus get 4 pieces of covering instead of 3 in *Case 1)*. In *Subcase 3″)* situation is more complicated, since we must take into account the position of the nodal point in the long cylinders — the "parts inbetween". Thus we must consider now the sequence of cylinders with one marked point, *i.e.* the sequence of pants. The fact that $R_n \longrightarrow 0$ and $\frac{r_n}{R_n} \longrightarrow 0$ means that the conformal structure of those pants is not constant and converges to one of the sphere with 3 punctures.

In order to have the covering pieces with the convergence types *A)–C)*, we choose an appropriate refinement of the covering. After it, we obtain 2 sequences of long cylinders, describing appearance of 2 new nodal points. The first one corresponds to the part between the "original" nodal point and bubbled sphere and is represented by V_5, whereas the other one, represented by V_2, lies on the other side from the "original" nodal point. Besides, we fix a neighborhood of the "original" nodal point which has a constant complex structure and is topologically an annulus with the disc Δ'' attached to the nodal point. To satisfy the requirements of *Theorem 4.2*, we cover the neighborhood by 2 pieces, the pants V_3 and the piece V_4 parametrizing the nodes $W_4^{(n)}$. This explains appearance of 6 pieces of covering in *Subcase 3″)*.

Lemma 4.5. *The limit curve (C_∞, u_∞) constructed in the proof of Theorem 1.1 is stable over X.*

Proof. If (C_∞, u_∞) is unstable over X, then either C_∞ is a torus with u_∞ constant, or C_∞ should have a component $C' \subset C_\infty$, such that $\tilde{C}'$ is a sphere with at most 2 marked points and $u_\infty(C')$ is a point.

The case of a constant map from a torus is easy to handle. In fact, in this case all C_n must be also tori with $\mathsf{area}\,(u_n(C_n))$ sufficiently small for $n >> 1$. Cover every C_n by infinite cylinder $Z(-\infty, +\infty)$ and consider compositions $\tilde{u}_n : Z(-\infty, +\infty) \to X$ of u_n with the covering maps. Since $\mathsf{area}\,(u_n(C_n)) \approx 0$, *Corollary 3.6* can be applied to show that every $\tilde{u}_n$ extends to a J_n-holomorphic map from S^2 to X. Consequently, $\mathsf{area}\,(\tilde{u}_n(Z(-\infty, +\infty)))$ must be finite. On the other hand, $\mathsf{area}\,(u_n(C_n)) > 0$ due to the stability condition, and hence $\mathsf{area}\,(\tilde{u}_n(Z(-\infty, +\infty)))$ must be infinite. This contradiction rules out the case of a torus.

The same argumentations go through in the case, when C_∞ is the sphere with no marked points. Then the curves C_n are also parametrized by the sphere S^2. The condition of instability means that $\mathsf{area}\,(u_\infty(C_\infty)) = 0$. Due to *Corollary 3.3*, $\mathsf{area}\,(u_n(C_n))$ must be sufficiently small for $n >> 1$. Now *Lemma 3.8* and the stability of (C_n, u_n) show that this is im possible.

Now consider the cases when the limit curve C_∞ has a "bubbled" component C', which is the sphere with 1 or 2 marked points. If C' has 1 marked point, then C' must appear as a "bubbled" sphere (S^2, v_∞) in the constructions of *Cases 1)–3)* in the proof of *Theorem 1.1*. But these constructions yield only non-trivial "bubbled" spheres, for which $v_\infty \neq const$. Thus such component C' must be stable.

In the remaining case, *i.e.* if there exists a component C' with 2 marked points, we consider a domain $U \subset C_\infty$, which is the union of the component C' and neighborhoods of the marked point on C'. If C' is an unstable component, then $\mathsf{area}\,(u_\infty(C')) = 0$ and we can achieve the estimate $\mathsf{area}\,(u_\infty(U)) < \varepsilon$ taking U sufficiently small. Set $\Omega := \sigma_\infty^{-1}(U)$, where $\sigma_\infty : \Sigma \to C_\infty$ is the parametrization of C_∞. Let γ_1 and γ_2 be the pre-images of marked points on C'. Then Ω must be a topological annulus, and γ_i, $i = 1, 2$, disjoint circles generating the group $\pi_1(\Omega) = \mathbb{Z}$. Further, C' must be a "bubbling" component of C_∞, *i.e.* at least for one of the circles γ_i, $i = 1, 2$, the images $\sigma_n(\gamma_i)$ are not nodal points of C_n but smooth circles.

If the both circles γ_1 and γ_2 are of this type, then $U_n := \sigma_n(\Omega)$ satisfies the conditions of *Lemma 3.7*. In this case we should have the convergence type *A)*, and hence the limit piece $\sigma_\infty(\Omega)$ should be isomorphic to the node $\mathcal{A}_0$.

In the case when only one circle, say γ_1, corresponds to nodal points on C_n, and for the other one the images $\sigma_n(\gamma_2)$ are smooth circles, then the domains $\sigma_n(\Omega)$ must be isomorphic to the node $\mathcal{A}_0$. Furthermore, due to the condition $\mathsf{area}\,(u_\infty(\sigma_\infty(\Omega))) < \varepsilon$ we have $\mathsf{area}\,(u_n(\sigma_n(\Omega))) < \varepsilon$. Consequently, we must have the convergence type *B)* and the unstable component C' could not appear. □

5. Curves with boundary on totally real submanifolds

In this section we consider the behavior of pseudoholomorphic curves over an almost complex manifold (X, J) with boundary on totally real submanifold(s). As in the "interior" case, we need to allow some type of boundary singularity.

Definition 5.1. *The set* $\mathcal{A}^+ := \{(z_1, z_2) \in \Delta^2 \,:\, z_1 \cdot z_2 = 0, \mathsf{Im}\, z_1 \geqslant 0, \mathsf{Im}\, z_2 \geqslant 0\}$ *is called the standard boundary node. A curve* $\overline{C}$ *with boundary* ∂C *is called a nodal curve with boundary if:*

i) C *is a nodal curve, possibly disconnected;*

ii) $\overline{C} = C \cup \partial C$ *is connected and compact;*

iii) every boundary point $a \in \partial C$ *has a neighborhood homeomorphic either to the half-disk* $\Delta^+ := \{z \in \Delta \,:\, \mathsf{Im}\, z \geqslant 0\}$, *or to the standard boundary node* $\mathcal{A}^+$.

In the last case $a \in \partial C$ *is called a boundary nodal point, whereas nodal points of* C *are called interior nodal points. Both boundary and interior nodal points are simply called nodal points.*

Definition 5.2. *Let* (X, J) *be an almost complex manifold. A pair* $(\overline{C}, u)$ *is called a curve with boundary over* (X, J) *if* $\overline{C} = C \cup \partial C$ *is a nodal curve with boundary, and* $u : \overline{C} \to (X, J)$ *is a continuous* $L^{1,2}$*-smooth map, which is pseudoholomorphic on* C.

A curve (C, u) with boundary is stable if the same condition as in *Definition 1.5* on the automorphism groups of compact irreducible components is satisfied.

Remark. One can see, that $\overline{C}$ has a uniquely defined real analytic structure, such that the normalization $\overline{C}^{\mathsf{nr}}$ is a real analytic manifold with boundary. More precisely, the pre-image of every boundary nodal point a_i consists of two points a_i' and a_i''. The normalization map $s : \overline{C}^{\mathsf{nr}} \to \overline{C}$ glues each pair a_i', a_i'') of points on $\overline{C}^{\mathsf{nr}}$ together into nodal points $a_i = s(a_i') = s(a_i'')$ on $\overline{C}$. This implies that the notion of an $L^{1,p}$-smooth map, $p > 2$, and that of also a continuous $L^{1,p}$-smooth map $u : \overline{C} \to X$ re well defined.

Definition 5.3. *We say that a real oriented surface with boundary $(\Sigma, \partial\Sigma)$ parameterizes a nodal curve with boundary C if there is a continuous map $\sigma : \overline{\Sigma} \to \overline{C}$ such that:*

i) if $a \in C$ is an interior nodal point, then $\gamma_a := \sigma^{-1}(a)$ is a smooth imbedded circle in Σ;

ii) if $a \in \partial C$ is a boundary nodal point, then $\gamma_a := \sigma^{-1}(a)$ is a smooth imbedded arc in Σ with end points on $\partial\Sigma$, transversal to $\partial\Sigma$ at these points;

iii) if $a, b \in \overline{C}$ are distinct (interior or boundary) nodal points, then $\gamma_a \cap \gamma_b = \varnothing$;

iv) $\sigma : \overline{\Sigma} \backslash \bigcup_{i=1}^{N} \gamma_{a_i} \to \overline{C} \backslash \{a_1, \ldots, a_N\}$ is a diffeomorphism, where $a_1, \ldots, a_N$ are all (interior and boundary) nodal points of $\overline{C}$.

Recall that a real subspace W of a complex vector space is called *totally real* if $W \cap iW = 0$. Similarly, a C^1-immersion $f : W \to X$ is called *totally real* if for any $w \in W$ the image $df(T_w W)$ is a totally real subspace of $T_{f(w)} X$.

Let $(\overline{C}, u)$ be a stable curve with boundary over an almost complex manifold (X, J) of a complex dimension n.

Definition 5.4. *We say that $(\overline{C}, u)$ satisfies totally real boundary condition $\boldsymbol{W}$ of type $\boldsymbol{\beta}$ if*

i) $\boldsymbol{\beta} = \{\beta_i\}$ is a collection of arcs with disjoint interiors, which defines a decomposition of the boundary $\partial C = \cup_i \beta_i$; moreover, we assume that every boundary nodal point is an endpoint for 4 arcs β_i;

ii) $\boldsymbol{W} = \{(W_i, f_i)\}$ is a collection of totally real immersions $f_i : W_i \to X$, one for every β_i;

iii) there are given continuous maps $u_i^{(b)} : \beta_i \to W_i$ realizing conditions $\boldsymbol{W}$, i.e. $f_i \circ u_i^{(b)} = u|_{\beta_i}$.

Remarks. 1. We shall consider (immersed) totally real submanifolds only of *maximal real dimension* $n = \mathsf{dim}_{\mathbb{C}} X$.

2. If $\boldsymbol{\beta}$ is a collection of arcs as above, a parametrization $\sigma : \overline{\Sigma} \to \overline{C}$ induces a collection of arcs $\sigma^{-1}(\boldsymbol{\beta}) := \{\sigma^{-1}(\beta_i) \ : \ \beta_i \in \boldsymbol{\beta}$ with the properties similar to *i)* of *Definition 5.4*. Thus, $\sigma^{-1}(\beta_i)$ have disjoint interiors, $\cup_i \sigma^{-1}(\beta_i) = \partial\Sigma$, and for any boundary node $a \in \overline{C}$ every endpoint of the arc $\beta_a = \sigma^{-1}(a)$ is an endpoint of two arcs $\sigma^{-1}(\beta_i)$. Since $\boldsymbol{\beta}$ is completely determined by $\sigma^{-1}(\boldsymbol{\beta})$, we shall denote the both collections simply by $\boldsymbol{\beta}$ and shall not distinguish them considering boundary conditions.

A totally real boundary condition is a suitable elliptic boundary condition for an elliptic differential operator $\overline{\partial}$ of Cauchy-Riemann type. In particular, all statements about “inner” regularity and convergence for pseudoholomorphic curves remain valid near “totally real” boundary. As in “inner” case, to get some “uniform” estimate at boundary one needs W to be “uniformly totally real”.

Definition 5.5. *Let X be a manifold with a Riemannian metric h, J a continuous almost complex structure, W a manifold, and $A_W \subset W$ a subset. We say that an immersion $f : W \to X$ is h-uniformly totally real along A_W with a lower angle $\alpha = \alpha(W, A_W, f) > 0$, iff*

i) $df : TW \to TX$ is h-uniformly continuous along A_W;

ii) for any $w \in A_W$ and any $\xi \neq 0 \in T_w W$ the angle $\angle_h\big(J df(\xi), df(T_w W)\big) \geqslant \alpha$.

We start with a generalization of the First Apriori Estimate. Define the half-disks $\Delta^+(r) := \{z \in \Delta(r) \,:\, \mathsf{Im}\, z \geqslant 0\}$ with $\Delta^+ = \Delta^+(1)$ and $\Delta^- := \{z \in \Delta \,:\, \mathsf{Im}\, z \leqslant 0\}$. Set $\beta_0 := (-1,1) \subset \partial\Delta^+$. Let X be a manifold with a Riemannian metric h, $A \subset X$ a subset, J^* a continuous almost complex structure, $f : W \to X$ a totally real immersion, and $A_W \subset W$ a subset.

Lemma 5.1. *Suppose that J^* is h-uniformly continuous on A with the uniform continuity modulus μ_{J^*}, and that $f : W \to X$ is h-uniformly totally real along A_W with a lower angle $\alpha_f > 0$ and the uniform continuity modulus μ_f. Then for every $2 < p < \infty$ there exists an $\varepsilon_1^b = \varepsilon_1^b(\mu_{J^*}, \alpha_f, \mu_f)$ (independent of p) and $C_p = C(p, \mu_{J^*}, \alpha_f, \mu_f)$, such that the following holds:*

If J is a continuous almost complex structure on X with $\|J - J^*\|_{L^\infty(A)} < \varepsilon_1^b$, if $u \in C^0 \cap L^{1,2}(\overline{\Delta}^+, X)$ is J-holomorphic map with $u(\Delta) \subset A$ and with the boundary condition $u|_{\beta_0} = f \circ u^b$ for some continuous $u^b : \beta_0 \to A_W \subset W$, then the condition $\|du\|_{L^2(\Delta^+)} < \varepsilon_1^b$ implies the estimate

$$\|du\|_{L^p(\Delta^+(\frac{1}{2}))} \leqslant C_p \cdot \|du\|_{L^2(\Delta^+)}. \tag{5.1}$$

Proof. Suppose additionally that $\mathsf{diam}(u(\Delta^+))$ is sufficiently small. Then we may assume that $u(\Delta^+)$ is contained in some chart $U \subset \mathbb{C}^n$, such that $\|J - J_{\mathrm{st}}\|_{L^\infty(U)}$ is also small enough. Let $z = (z_1, \dots, z_n)$ be J_{st}-holomorphic coordinates in U, such that $u(0) = \{z_i = 0\}$. Making an appropriate diffeomorphism of U, we may additionally assume that $W_0 := f(W) \cap U$ lies in $\mathbb{R}^n$ and that $J = J_{\mathrm{st}}$ along W_0.

Consider the trivial bundle $E := \Delta \times \mathbb{C}^n$ over Δ with complex structures J_{st} and $J_u := J \circ u$. We can consider u as a section of E over Δ^+ satisfying equation $\overline{\partial}_{J_u} u := \partial_x u + J_u \partial_y u = 0$. Over β_0 we get a J_u-totally real subbundle $F := \beta_0 \times \mathbb{R}^n$, such that $u(\beta_0) \subset F$. Let τ denote a complex conjugation in Δ and also a complex conjugation in E with respect to J_{st}. Extend J_u on $E|_{\Delta^-}$ as the composition $-\tau \circ J_u \circ \tau$. This means that for $z \in \Delta^-$ and we get

$$J_u(z) : v \mapsto \tau v \in E_{\tau z} \mapsto J_u(\tau z)(\tau v) \in E_{\tau z} \mapsto -\tau J_u(\tau z)(\tau v) \in E_z. \tag{5.2}$$

Since $J_u = J \circ u$ coincides with J_{st} along β_0, this extension is also continuous. Further, for $v \in L^1(\Delta^+, E)$ define the extension $\mathsf{ex}(v)$ by setting $v(z) := \tau v(\tau z)$. This gives continuous linear operators $\mathsf{ex} : L^p(\Delta^+, E) \to L^p(\Delta^+, E)$ for any $p \in [1, \infty]$. An important property of operator ex is that if $v \in L^{1,p}(\Delta^+, E)$ with $1 \leqslant p \leqslant \infty$ (resp. $v \in C^0(\overline{\Delta}^+)$) satisfies the boundary condition $v|_{\beta_0} \subset F$, then $\mathsf{ex}\, v \in L^{1,p}(\Delta, \mathbb{C}^n)$ (resp. $\mathsf{ex}\, v \in C^0(\overline{\Delta})$). Let us denote by $L^{1,p}(\Delta^+, E, F)$ (resp. by $C^0(\Delta^+, E, F)$) the spaces of all such v with the boundary condition $v|_{\beta_0} \subset F$.

Since $\partial_x(\tau v) = \tau(\partial_x v)$ and $\partial_y(\tau v) = -\tau(\partial_y v)$ for $v \in L^{1,1}(\Delta^+, E)$, we get $\overline{\partial}_{J_u}(\mathsf{ex}\, v) = \mathsf{ex}(\overline{\partial}_{J_u} v)$ for any $v \in L^{1,p}(\Delta^+, E, F)$. In particular, for $\tilde{u} := \mathsf{ex}\, u \in C^0 \cap L^{1,2}(\Delta, E)$ we have $\overline{\partial}_{J_u} \tilde{u} = 0$.

Starting from this point we can repeat the steps of the proof of *Lemma 3.1*.
□

Remark. We shall refer to the construction of a complex structure J_u in E over Δ^- and (resp. of a section $\tilde{u}$ of E over Δ^-) as *extension of $J \circ u$ (resp. of u) by the reflection principle*.

Let X be a manifold with a Riemannian metric h, J^* a continuous almost complex structures on X, $A \subset X$ a closed h-complete subset, such that J^* is h-uniformly continuous on A, and $f_0 : W \to X$ an immersion, which is h-uniformly totally real on some close f^*h-complete subset $A_W \subset W$.

Corollary 5.2. *Let $\{J_n\}$ be a sequence of continuous almost complex structures on X such that J_n converge h-uniformly on A to J, and $f_n : W \to X$ a sequence of totally real immersion such that df_n converge h-uniformly on A_W to df_0.*

Furthermore, let $u_n \in C^0 \cap L^{1,2}(\Delta^+, X)$ be a sequence of J_n-holomorphic maps, such that $u_n(\Delta^+) \subset A$, $\|du_n\|_{L^2(\Delta^+)} \leqslant \varepsilon_1^b$, $u_n(0)$ is bounded in X, and $u_n|_{\beta_0} = f_n \circ u_n^b$ for some continuous $u_n^b : \beta_0 \to A_W$.

Then there exists a subsequence u_{n_k} which $L^{1,p}_{\mathrm{loc}}(\Delta^+)$-converges to a J-holomorphic map u_∞ for all $p < \infty$.

Here $\beta_0 = (-1,1) \subset \partial\Delta^+$ and $L^{1,p}_{\mathrm{loc}}(\Delta^+)$-convergence means $L^{1,p}(\Delta^+(r))$-convergence for all $r < 1$, *i.e.* convergence up to boundary component β_0.

Proof. The statement is a generalization of *Corollary 3.3.* The proof of that statement goes through with an appropriate modification using the reflection principle.

Consider now a generalization of the Second Apriori Estimate. Instead of "long cylinders" we now have "long strips" satisfying appropriate boundary conditions.

Definition 5.5. *Define a strip $\Theta(a,b) := (a,b) \times [0,1]$ with the complex coordinate $\zeta := t - i\theta$, $t \in (a,b)$, $\theta \in [0,1]$. Define also $\Theta_n := \Theta(n-1,n)$, $\partial_0\Theta(a,b) := (a,b) \times \{0\}$, and $\partial_1\Theta(a,b) := (a,b) \times \{1\}$.*

We are interested in maps $u : \Theta(a,b) \to X$, which are holomorphic with respect to the complex coordinate ζ on $\Theta(a,b)$ and a continuous almost complex structure J on X, and which satisfy boundary conditions

$$u\big|_{\partial_0\Theta(a,b)} = f_0 \circ u_0^b, \qquad\qquad u\big|_{\partial_1\Theta(a,b)} = f_1 \circ u_1^b,$$

with some J-totally real immersions $f_{0,1} : W_{0,1} \to X$ and continuous maps $u_{0,1} : \partial_{0,1}\Theta(a,b) \to W_{0,1}$. First we consider the linear case.

Lemma 5.3. *Let W_0 and W_1 be n-dimensional totally real subspaces in $\mathbb{C}^n = (\mathbb{R}^n, J_{\mathrm{st}})$. Then there exist a constant $\gamma_W = \gamma(n, W_0, W_1)$ with $0 < \gamma_W < 1$ such that for any holomorphic map $u : \Theta(0,3) \to \mathbb{C}^n$ with the boundary conditions*

$$u(\partial_0\Theta(0,3)) \subset W_0 \qquad u(\partial_1\Theta(0,3)) \subset W_1 \tag{5.2}$$

we have the following estimate:

$$\int_{\Theta_2} |du|^2 dt\, d\theta \leqslant \frac{\gamma_W}{2}\left(\int_{\Theta_1} |du|^2 dt\, d\theta + \int_{\Theta_3} |du|^2 dt\, d\theta\right). \tag{5.3}$$

Proof. Let $L^{1,2}_W([0,1],\mathbb{C}^n)$ be a Banach manifold $v(\theta) \in L^{1,2}([0,1],\mathbb{C}^n)$, such that $v(0) \in W_0$ and $v(1) \in W_1$. Consider a nonnegative quadratic form $Q(v) := \int_0^1 |\partial_\theta v(\theta)|^2 d\theta$. Since $Q(v) + \|v\|^2_{L^2} = \|v\|^2_{L^{1,2}}$ and the imbedding $L^{1,2}_W([0,1],\mathbb{C}^n) \hookrightarrow L^2([0,1],\mathbb{C}^n)$ is compact, we can decompose $L^{1,2}_W([0,1],\mathbb{C}^n)$ into a direct Hilbert sum of eigenspaces $\mathbb{E}_\lambda$ of Q w.r.t. $\|v\|^2_{L^2}$. This means that v_λ belongs to $\mathbb{E}_\lambda$ iff

$$\int_0^1 \langle \partial_\theta v_\lambda(\theta), \partial_\theta w(\theta)\rangle d\theta = \int_0^1 \lambda \langle v_\lambda(\theta), w(\theta)\rangle d\theta, \tag{5.4}$$

for any $w \in L^{1,2}_W([0,1],\mathbb{C}^n)$. Here $\langle\cdot,\cdot\rangle$ denotes a standard $\mathbb{R}$-*valued* scalar product in $\mathbb{C}^n$. Integrating by parts yields

$$\int_0^1 \langle \partial^2_{\theta\theta} v_\lambda(\theta) + \lambda v_\lambda(\theta), w(\theta)\rangle d\theta + \langle \partial_\theta v_\lambda(\theta), w(\theta)\rangle|_{\theta=1} - \langle \partial_\theta v_\lambda(\theta), w(\theta)\rangle|_{\theta=0} = 0.$$

This implies that v_λ belongs to $\mathbb{E}_\lambda$ *iff* $\partial^2_{\theta\theta} v_\lambda(\theta) + \lambda v_\lambda(\theta) = 0$, $\partial_\theta v_\lambda(1) \perp W_1$, and $\partial_\theta v_\lambda(0) \perp W_0$. Since $J_{\rm st}$ is $\langle\cdot,\cdot\rangle$-orthogonal, we can conclude that $J\partial_\theta v_\lambda(\theta) \in \mathbb{E}_\lambda$.

Positivity and compactness of Q w.r.t. $\|\cdot\|_{L^2}$ imply that all $\mathbb{E}_\lambda$ are finite dimensional and $\mathbb{E}_\lambda = \{0\}$ for $\lambda < 0$. Further, since $\partial_\theta v = 0$ for any $v \in \mathbb{E}_0$, the space $\mathbb{E}_0$ consists of constant functions with values in $W_0 \cap W_1$.

Now let $u : \Theta(0,3) \to \mathbb{C}^n$ be a holomorphic map with the boundary condition (5.2). We can represent u in the form $u(t,\theta) = \sum_\lambda u_\lambda(t,\theta)$ with $u_\lambda(t,\cdot) := \mathsf{pr}_\lambda(u(t,\cdot)) \in L^{1,2}([0,3],\mathbb{E}_\lambda)$. Since $J\partial_\theta$ is an endomorphism of every $\mathbb{E}_\lambda$, every $u_\lambda(t,\theta)$ is also holomorphic. In particular, u_0 is also holomorphic and constant in θ. Thus u_0 is constant.

Since u is harmonic, $(\partial^2_{tt} + \partial^2_{\theta\theta})u = 0$, we get $\partial^2_{tt} u_\lambda(t,\theta) = \lambda u_\lambda(t,\theta)$. For $\lambda > 0$ this yields $u_\lambda(t,\theta) = e^{+\sqrt{\lambda}t} v^+_\lambda(\theta) + e^{-\sqrt{\lambda}t} v^-_\lambda(\theta)$ with $v^\pm_\lambda(\theta) \in \mathbb{E}_\lambda$. Fixing an orthogonal $\mathbb{R}$-basis of $\mathbb{E}_\lambda$ v^i_λ we write every u_λ in the form

$$u_\lambda(t,\theta) = \sum_i (a^i_\lambda e^{+\sqrt{\lambda}t} + b^i_\lambda e^{-\sqrt{\lambda}t}) v^i_\lambda(\theta)$$

with real constants a^i_λ, b^i_λ. Since $u_0(t,\theta)$ is constant, $\|du\|^2_{L^2(\Theta_k)} = 2\|\partial_\theta u\|^2_{L^2(\Theta_k)} = \sum_{\lambda,i} 2\lambda \int_{k-1}^k (a^i_\lambda e^{+\sqrt{\lambda}t} + b^i_\lambda e^{-\sqrt{\lambda}t})^2 d\theta$. Here we use (5.4) and L^2-orthonormality of v^i_λ. This leads us to the problem of finding the smallest possible constant γ in the inequality

$$\int_1^2 (ae^{\alpha t} + be^{-\alpha t})^2 dt \leqslant \frac{\gamma}{2}\left(\int_0^1 (ae^{\alpha t} + be^{-\alpha t})^2 dt + \int_2^3 (ae^{\alpha t} + be^{-\alpha t})^2 dt\right) \tag{5.5}$$

with $a, b \in \mathbb{R}$ for given $\alpha > 0$. The integration gives

$$a^2 e^{3\alpha}\frac{e^\alpha - e^{-\alpha}}{2\alpha} + b^2 e^{-3\alpha}\frac{e^\alpha - e^{-\alpha}}{2\alpha} + 2ab \leqslant$$

$$\leqslant \frac{\gamma}{2}\left(a^2 e^{3\alpha}\frac{(e^\alpha - e^{-\alpha})(e^{2\alpha} + e^{-2\alpha})}{2\alpha} + b^2 e^{-3\alpha}\frac{(e^\alpha - e^{-\alpha})(e^{2\alpha} + e^{-2\alpha})}{2\alpha} + 4ab\right)$$

or, equivalently,

$$a^2 e^{3\alpha}\frac{(e^\alpha - e^{-\alpha})(e^{2\alpha} + e^{-2\alpha} - 2/\gamma)}{2\alpha} + b^2 e^{-3\alpha}\frac{(e^\alpha - e^{-\alpha})(e^{2\alpha} + e^{-2\alpha} - 2/\gamma)}{2\alpha}$$

$$+4ab(1 - 1/\gamma) \geqslant 0$$

The determinant of the last quadratic form in a, b is

$$\left(\frac{(e^\alpha - e^{-\alpha})(e^{2\alpha} + e^{-2\alpha} - 2/\gamma)}{2\alpha}\right)^2 - 4(1 - 1/\gamma)^2 = 4\left(\frac{\mathsf{sh}\alpha(\mathsf{ch}\, 2\alpha - 1/\gamma)}{\alpha}\right)^2 - 4(1 - 1/\gamma)^2$$

$$\geqslant 4(\mathsf{ch}\, 2\alpha - 1/\gamma)^2 - 4(1-1/\gamma)^2 = 4(\mathsf{ch}\, 2\alpha - 1)(\mathsf{ch}\, 2\alpha + 1 - 2/\gamma).$$

Thus inequality (5.5) holds for every $a, b \in \mathbb{R}$ provided $\gamma \geqslant \frac{2}{1+\mathsf{ch}\, 2\alpha} < 1$.

Note that the minimal positive eigenvalue $\lambda_1 > 0$ of Q exists. Thus we can set $\gamma_W := \frac{2}{1+\mathsf{ch}(2\sqrt{\lambda_1})} < 1$ in estimate (5.3). □

Remark. A behavior of γ_W as a function of $\lambda_1 = \lambda_1(W_0, W_1)$ shows that $\gamma_W < 1$ can be chosen the same for all pairs $(\widetilde{W}_0, \widetilde{W}_1)$ sufficiently close to (W_0, W_1) provided $\mathsf{dim}\,(\widetilde{W}_0 \cap \widetilde{W}_1) = \mathsf{dim}\,(W_0 \cap W_1)$. Vice versa, if we perform sufficiently small deformation of (W_0, W_1) into $(\widetilde{W}_0, \widetilde{W}_1)$ with $\mathsf{dim}\,(\widetilde{W}_0 \cap \widetilde{W}_1) < \mathsf{dim}\,(W_0 \cap W_1)$, then some $v \in \mathbb{E}_0(W_0, W_1)$ will become an eigenvector $\tilde{v} \notin \mathbb{E}_0(\widetilde{W}_0, \widetilde{W}_1) = \widetilde{W}_0 \cap \widetilde{W}_1$, but with sufficiently small eigenvalue $\lambda_1(\widetilde{W}_0, \widetilde{W}_1) > 0$, so that the best possible $\gamma_{\widetilde{W}}$ will be arbitrary close to 1. Thus the uniform separation of γ_W from 1 under small perturbation of (W_0, W_1) is equivalent to uniform separation of $\lambda_1(W_0, W_1)$ from 0, which is equivalent to constancy of $\mathsf{dim}\,(W_0 \cap W_1)$.

Another phenomenon, also connected with spectral behavior, is that for *harmonic* $u(t,\theta)$, $\mathbb{E}_0 = \{const\}$ does not imply $u_0 = const$, but merely $u_0(t,\theta) = v_0 + v_1 t$ with $v_0, v_1 \in \mathbb{E}_0$, so that inequality (5.3) in not true. This leads to much more complicated bubbling with energy loss for harmonic and harmonic-type maps, compare [S-U], [P-W], [Pa].

Note also that if W_0 and W_1 are *affine* totally real subspaces of $\mathbb{C}^n$, then inequality (5.3) for holomorphic $u : \Theta(0,3) \to \mathbb{C}^n$ with boundary condition (5.2) is in general not true. An easy example is a natural imbedding $u : \Theta(0,3) \hookrightarrow \mathbb{C}$, with non-constant component $u_0 \equiv u$ and with $\int_{\Theta(0,1)} |du|^2 = \int_{\Theta(1,2)} |du|^2 = \int_{\Theta(2,3)} |du|^2 \neq 0$. In general, those are n-dimensional totally real affine planes W_0 and W_1 in $\mathbb{C}^n$ with empty intersection. This can happen if W_0 and W_1 are parallel or *skew*. The later means that the corresponding vector spaces V_0 and V_1 ($W_i = V_i + w_i$ for some $w_i \in \mathbb{C}^n$) are different. In both cases the intersection $V_0 \cap V_1$ is not zero, since otherwise $W_0 \cap W_1 \neq \varnothing$ by dimension argumentation.

The considerations above show which properties should be controlled to obtain a reasonable statement in the nonlinear case.

Definition 5.6. *Let X be a manifold with a Riemannian metric h, $f_0 : W_0 \to X$ and $f_1 : W_1 \to X$ immersions, and $A_0 \subset W_0$, $A_1 \subset W_1$ subsets. We say that $f_0 : W_0 \to X$ and $f_1 : W_1 \to X$ are h-uniformly transversal along A_0 and A_1 with parameters $\delta > 0$ and M if for any $x_0 \in A_0$ and $x_1 \in A_1$ one of the following conditions hold:*

i) $\mathsf{dist}_h(f_0(x_0), f_1(x_1)) > \delta$;

ii) there exists $x_i' \in A_i$ with $f_0(x_0') = f_1(x_1')$ and such that

$$\mathsf{dist}_h(x_0, x_0') + \mathsf{dist}_h(x_1, x_1') \leqslant M\, \mathsf{dist}_h(f_0(x_0), f_1(x_1)).$$

Remark. Roughly speaking, the condition excludes appearance of points where W_0 and W_1 are "asymptotically parallel or skew" and ensures us existence of a uniform lower bound for the angle between W_0 and W_1.

Let now X be a manifold with a Riemannian metric h, J^* a continuous almost complex structure on X, $f_0 : W_0 \to X$ and $f_1 : W_1 \to X$ immersions, and $A \subset X$,

$A_0 \subset W_0$, $A_1 \subset W_1$ subsets. Suppose that J^* is h-uniformly continuous on A, $df_i : TW_i \to TX$ are h-uniformly continuous on A_i, and that f_i are h-uniformly transversal along A_i with parameters $\delta = \delta(f_0, f_1) > 0$ and $M = M(f_0, f_1)$.

Lemma 5.4. *There exist constants $\varepsilon_2^b = \varepsilon_2^b(\mu_{J^*}, f_0, f_1, \delta, M) > 0$ and $\gamma^b = \gamma^b(\mu_{J^*}, f_0, f_1, \delta, M) < 1$ such that for any continuous almost complex $\tilde J$ with $\|\tilde J - J^*\|_{L^\infty(A)} < \varepsilon_2^b$, any immersions $\tilde f_i : W_i \to X$ with $\mathsf{dist}(\tilde f_i, f_i)_{C^1(A_i)} < \varepsilon_2^b$, and any $\tilde J$-holomorphic map $u \in C^0 \cap L^{1,2}(\Theta(0,5), X)$ with $u(\Theta(0,5)) \subset A$, $u|_{\partial_i\Theta(0,5)} = f_i \circ u_i^b$ for some continuous $u_i^b : \partial_i\Theta(0,5) \to A_i \subset W_i$ the conditions*
i) $\|du\|_{L^2(\Theta_i)} < \varepsilon_2^b$;
ii) $\tilde f_i : W_i \to X$ *are h-uniformly transversal along A_i with the same parameters δ and M*

imply the estimate

$$\|du\|^2_{L^2(\Theta_3)} \leqslant \frac{\gamma^b}{2} \cdot \Big(\|du\|^2_{L^2(\Theta_2)} + \|du\|^2_{L^2(\Theta_4)} \Big).$$

Proof. Suppose the statement of the lemma is false. Then there should exist a sequence of continuous almost complex structures J_k with $\|J_k - J^*\|_{L^\infty(A)} \longrightarrow 0$, a sequence of immersions $f_{k,i} : W_i \to X$ with $f_{k,i} \longrightarrow f_i$ in $C^1(A_i)$, such that $f_{k,i} : W_i \to X$ are h-uniformly transversal with the same parameters δ and M, and a sequence of J_k-holomorphic maps $u_k \in C^0 \cap L^{1,2}(\Theta(0,5), X)$ with $u_k(\Theta(0,5)) \subset A$ and $u_k|_{\partial_i\Theta(0,5)} = f_{k,i} \circ u^b_{n,i}$ for some continuous $u^b_{n,i} : \partial_i\Theta(0,5) \to A_i \subset W_i$, such that $\|du_k\|^2_{L^2(\Theta(0,5))} \longrightarrow 0$ and

$$\|du_k\|^2_{L^2(\Theta_3)} \geqslant \frac{\gamma_k}{2} \cdot \Big(\|du_k\|^2_{L^2(\Theta_2)} + \|du_k\|^2_{L^2(\Theta_4)} \Big)$$

with $\gamma_k = 1 - 1/k$. *Lemmas 4.1* and *5.1* provide that in this case $\mathsf{diam}_h(u_k(\Theta(1,4))) \longrightarrow 0$.

Since $f_{k,i} : W_i \to X$ are h-uniformly transversal with the same parameters δ and M, there should exist sequences $x_k \in A$, $x_{k,0} \in A_0$, and $x_{k,1} \in A_1$ such that $x_k = u_0(x_{k,0}) = u_1(x_{k,1})$ and $u_k(\Theta(1,4)) \subset B(x_k, r_k)$ with $r_k \longrightarrow 0$. The h-uniform continuity of J^* implies that there exist C^1-diffeomorphisms $\varphi_k : B(x_k, r_k) \to B(0, r_k) \subset \mathbb{C}^n$ with $\|J_k - \varphi_k^* J_{\mathrm{st}}\|_{L^\infty(B(x_k,r_k))} + \|h - \varphi_k^* h_{\mathrm{st}}\|_{L^\infty(B(x_k,r_k))} \longrightarrow 0$.

Using φ_k we transfer our situation into $B(0, r_k) \subset \mathbb{C}^n$ and rescale it. Namely we set $\alpha_k := \|du_k\|_{L^2(\Theta_3)}$ and define diffeomorphisms $\psi_k := \frac{1}{\alpha_k} \circ \varphi_k : B(x_k, r_k) \to B(0, R_k) \subset \mathbb{C}^n$ with $R_k := \alpha_k^{-1} \cdot r_k$. Note that by *Lemmas 4.1* and *5.1* we have $\alpha_k = \|du_k\|_{L^2(\Theta(2,3))} \leqslant C \mathsf{diam}_h(u_k(\Theta(1,4))) \leqslant C' r_k$, so that R_k are uniformly bounded from below.

In $B(0, R_k)$ we consider Riemannian metrics $h_k := \alpha_k^{-2} \cdot \psi_{k*} h_k$ (*i.e.* pushed forward and α_k^{-2}-rescaled h_k), almost complex structures $J_k^* := \psi_{k*} J_k$, and J_k^*-holomorphic maps $u_k^* := \psi_k \circ u_k : \Theta(1,4) \to B(0, R_k)$. Note that here we consider h as a metric tensor, so multiplying h by α^{-2} we increase h-norms and h-distances in α^{-1} and not in α^{-2} times.

Then $\|du_k^*\|_{L^2(\Theta_3, h_k)} = 1$, $\|du_k^*\|^2_{L^2(\Theta_2, h_k)} + \|du_k^*\|^2_{L^2(\Theta_4, h_k)} \leqslant \frac{2k}{k-1}$, and

$$\|J_k^* - J_{\mathrm{st}}\|_{L^\infty(B(0,R_k), h_k)} = \|J_k - \varphi_k^* J_{\mathrm{st}}\|_{L^\infty(B(x_k,r_k), h)} \longrightarrow 0.$$

The last equality uses an obvious relation

$$\frac{|F(\xi)|_{\alpha^{-2}\cdot h}}{|\xi|_{\alpha^{-2}\cdot h}} = \frac{\alpha^{-1}\cdot|F(\xi)|_h}{\alpha^{-1}\cdot|\xi|.h} = \frac{|F(\xi)|_h}{|\xi|_h}$$

for any linear $F:T_xX\to T_xX$ and $\xi\neq 0\in T_xX$. In a similar way, we also obtain $\|h_k-h_{\rm st}\|_{L^\infty(B(0,R_k),h_k)}\longrightarrow 0$.

Going to a subsequence, we may additionally assume that the tangent spaces $d\psi_k\circ df_i(T_{x_{k,i}}W_i)$, $i=1,2$, converge to some spaces $W_i^*\subset\mathbb{C}^n$. Since W_i are uniformly totally real, W_i^* are also totally real linear subspaces in $\mathbb{C}^n$. Since the maps $df_i:TW_0\to TX$ are uniformly continuous on $A_i\subset W_i$, $f_{k,i}\longrightarrow f_i$ in $C^1(A_i)$, and since $r_k\longrightarrow 0$, the images $W_{k,i}^*:=\psi_k\circ f_{k,i}(B_{W_i}(x_{k,i},r_k))$ of the balls $B_{W_i}(x_{k,i},r_k)\subset W_i$ are imbedded submanifolds of $\mathbb{C}^n$ with $0\in W_{k,i}^*$, which converge to W_i^* in Hausdorff topology. Moreover, we can consider $W_{k,i}^*$ as graphs of maps $g_{k,i}$ from subdomains $U_{k,i}\subset W_i^*\cap B(0,R_k)$ to $W_i^{*\perp}$ and for any fixed $R\leqslant\inf\{R_k\}$ the restrictions $g_{k,i}|_{W_i^*\cap B(0,R)}$ converge to zero map from $W_i^*\cap B(0,R)$ to $W_i^{*\perp}$.

The apriori estimates for the maps $u_k^*:\Theta(1,4)\to\mathbb{C}^n$ imply that for any $p<\infty$ the maps u_k^* converge in weak- $L^{1,p}$-topology to some $J_{\rm st}$-holomorphic map $u^*:\Theta(1,4)\to\mathbb{C}^n$. Further, since u_k^* satisfy totally real boundary conditions $u_k^*|_{\partial_i\Theta(1,4)}\subset W_{k,i}^*$, the same is true for u^*, *i.e.* $u^*|_{\partial_i\Theta(1,4)}\subset W_i^*$. Nice behavior of $W_{k,i}^*$ shows that on Θ_3 we have also a strong convergence, and hence $\|du^*\|_{L^2(\Theta_3)}=\lim\|du_k^*\|_{L^2(\Theta_3)}=1$. In particular, u^* is not constant. On the other hand, $\|du^*\|^2_{L^2(\Theta_2)}+\|du^*\|^2_{L^2(\Theta_4)}\leqslant\lim\|du_k^*\|^2_{L^2(\Theta_2)}+\|du_k^*\|^2_{L^2(\Theta_4)}\leqslant 2$. The obtained contradiction with *Lemma 5.3* shows that *Lemma 5.4* is true. □

Let X, h, J, A, $f_i:W_i\to X$, A_i, and the constant ε_2^b and γ^b be as in *Lemma 5.4*. Suppose that $\tilde J$ is a continuous almost complex structure on X with $\|\tilde J-J\|_{L^\infty(A)}<\varepsilon_2^b$, $\tilde f_i:W_i\to X$ are totally real immersions with $\mathsf{dist}(\tilde f_i,f_i)_{C^1(A_i)}\leqslant\varepsilon_2^b$, such that f_i are h-uniformly transversal along A_i with the same parameters δ and M as $f_i:W_i\to X$.

Corollary 5.5. *Let $u\in C^0\cap L^{1,2}(\Theta(0,l),X)$ be a $\tilde J$-holomorphic map, such that $u(\Theta(0,l))\subset A$, $u|_{\partial_i\Theta(0,l)}=\tilde f_i\circ u_i^b$ for some continuous $u_i^b:\partial_i\Theta(0,l)\to A_i\subset W_i$, and such that $\|du\|_{L^2(Z_k)}<\varepsilon_2$ for any $k=1,\dots,l$.*

Let $\lambda_b>1$ be the (uniquely defined) real number with $\lambda_b=\frac{\gamma^b}{2}(\lambda_b^2+1)$. Then for $2\leqslant k\leqslant l-1$ one has

$$\|du\|^2_{L^2(\Theta_k)}\leqslant\lambda_b^{-(k-2)}\cdot\|du\|^2_{L^2(\Theta_2)}+\lambda_b^{-(l-1-k)}\cdot\|du\|^2_{L^2(\Theta_{n-1})}. \tag{5.?9}$$

Proof. The same as in *Lemma 4.5*. □

Immediate corollary of this estimate is a lower bound of energy on nonconstant "infinite strip".

Lemma 5.6. *In the setting of Corollary 5.5, let $u\in C^0\cap L^{1,2}(\Theta(-\infty,+\infty),X)$ be a nonconstant $\tilde J$-holomorphic map, such that $u(\Theta(-\infty,+\infty))\subset A$ and $u|_{\partial_i\Theta(0,l)}=\tilde f_i\circ u_i^b$ for some continuous $u_i^b:\partial_i\Theta(-\infty,+\infty)\to A_i\subset W_i$. Then $\|du\|_{L^2(\Theta_k)}>\varepsilon_2^b$ for some k. In particular, $\|du\|_{L^2(\Theta(-\infty,+\infty))}>\varepsilon_2^b$.*

Proof. *Corollary 5.5* shows that if $\|du\|_{L^2(\Theta_k)} \leqslant \varepsilon_2^b$ for all k, then $\|du\|_{L^2(\Theta_k)} = 0$, *i.e.* u is constant. □

Another consequence of *Corollary 5.5* is a generalization of the Gromov's result about removability of boundary point singularity, see [G]. An important improvement is the fact that the statement remains valid also when one has *different* boundary conditions on the left and on the right from a singular point. One can see such a point x as a *corner point* for the corresponding pseudoholomorphic curve. Typical examples appear in symplectic geometry where one takes Lagrangian submanifolds as boundary conditions.

Define the punctured half-disk by setting $\check\Delta^+ := \Delta^+\backslash\{0\}$. Define $I_- := (-1,0) \subset \partial\check\Delta^+$ and $I_+ := (0,+1) \subset \partial\check\Delta^+$.

Corollary 5.7. *(Removal of boundary point singularities). Let X be a manifold with a Riemannian metric h, J a continuous almost complex structure, $f_i : W_i \to X$, $i = 1,2$, totally real immersions, and $A_i \subset W_i$ subsets. Let $u : (\check\Delta^+, J_{\mathrm{st}}) \to (X, J)$ be a pseudoholomorphic map. Suppose that*

i) *J is uniformly continuous on $A := u(\check\Delta)$ w.r.t. h, and closure of A is h-complete;*
ii) *u satisfies boundary conditions of the form $u|_{I_+} = f_0 \circ u_+^b$ and $u|_{I_-} = f_1 \circ u_-^b$ with come continuous $u_+^b : I_+ \to A_0 \subset W_0$ and $u_-^b : I_- \to A_1 \subset W_1$;*
iii) *f_i are h-uniformly totally real on A_i and h-uniformly transversal along A_i;*
iv) *there exists k_0, such that for all half-annuli $R_k^+ := \{z \in \Delta^+ : \frac{1}{e^{\pi(k+1)}} \leqslant |z| \leqslant \frac{1}{e^{\pi k}}\}$ with $k \geqslant k_0$ one has $\|du\|^2_{L^2(R_k^+)} \leqslant \varepsilon_2^b$, ε_2^b being from Lemma 5.4.*

Then u extends to the origin $0 \in \Delta^+$ as an $L^{1,p}$-map for some $p > 2$.

Proof. Using the holomorphic map $\mathsf{exp} : \Theta(0,\infty) \to \check\Delta^+$, $\mathsf{exp}(\theta + it) := e^{\pi(-t+i\theta)}$, we can reduce our situation to the case of pseudoholomorphic map $u^* := u \circ \mathsf{exp}$ form "infinite strip" $\Theta(0,\infty)$. By *Corollary 5.5*, for $k \geqslant k_0$ we obtain estimate $\|du^*\|_{L^2(\Theta_k)} \leqslant \lambda_b^{-(k-k_0)/2}\|du^*\|_{L^2(\Theta_{k_0})}$ with some $\lambda_b > 1$. This is equivalent to the estimate $\|du\|_{L^2(R_k^+)} \leqslant \lambda_b^{-(k-k_0)/2}\|du\|_{L^2(R_{k_0}^+)}$. *Lemmas 4.1* and *5.1* and scaling property of L^p-norms provide the estimate

$$\|du\|_{L^p(R_k^+)} \leqslant Ce^{-k(\mathsf{log}\lambda_b/2+\pi(2/p-1))}$$

Thus $du \in L^p(\Delta^+)$ for any p with $\mathsf{log}\lambda_b/2 > \pi(1-2/p)$, which means $p < \frac{4\pi}{2\pi - \mathsf{log}\lambda_b}$. □

Remark. Unlike the "interior" and smooth boundary cases, it is possible that the map u as in *Corollary 5.7* is not $L^{1,p}$-regular in the neighborhood of "corner point" $0 \in \Delta^+$ for some $p > 2$. For example, the map $u(z) = z^\alpha$ with $0 < \alpha < 1$ satisfies totally real boundary conditions $u(I_+) \subset \mathbb{R}$, $u(I_-) \subset e^{\alpha\pi i}\mathbb{R}$ and is $L^{1,p}$-regular only for $p < p^* := \frac{2}{1-\alpha}$.

As in the "interior" case, for the proof of the boundary compactness theorem we shall need a description of a convergence of a sequence of "long strip". Let X be a manifold with a Riemannian metric h, J a continuous almost complex structure, $A \subset X$ a closed h-complete subset, such that J is h-uniformly continuous on A, and let $\{J_n\}$ be a sequence of almost complex structures converging h-uniformly

on A to J. Let also $f_0 : W_0 \to X$ and $f_1 : W_1 \to X$ be immersions, $A_i \subset W_i$ subsets, such that df_i are uniformly h-uniformly totally real on A_i and f_i are h-uniformly transversal along A_i. Let $f_{n,i} : W_i \to X$ be totally real immersions, which C^1-converge to f_i on A_i, such that $f_{n,0}$ and $f_{n,1}$ are h-uniformly transversal along A_i with uniform in n parameters δ and C^*. Finally, let $\{l_n\}$ be a sequence of integers with $l_n \to \infty$, and $u_n : \Theta(0,l_n) \to X$ a sequence of J_n-holomorphic maps, satisfying boundary conditions $u_n|_{\partial_i\Theta(0,l_n)} = f_{n,i} \circ u^b_{n,i}$ with some continuous $u^b_{n,i} : \partial_i\Theta(0,l_n) \to A_i \subset W_i$.

Lemma 5.8. *In the described situation, suppose additionally that $u_n(\Theta(0,l_n)) \subset A$ and $\|du_n\|_{L^2(\Theta_k)} \leqslant \varepsilon^b_2$ for all n and $k \leqslant l_n$. Take a sequence $k_n \to \infty$ such that $k_n < l_n - k_n \to \infty$. Then:*
1) $\|du_n\|_{L^2(\Theta(k_n,l_n-k_n))} \to 0$ and $\mathsf{diam}\big(u_n(\Theta(k_n,l_n-k_n))\big) \to 0$.
2) There is a subsequence $\{u_n\}$, still denoted $\{u_n\}$, such that both $u_n|_{\Theta(0,k_n)}$ and $u_n|_{\Theta(k_n,l_n)}$ converge in $L^{1,p}$-topology on compact subsets in $\check\Delta^+ \cong \Theta(0,+\infty)$ to a J^*-holomorphic maps u^-_∞ and u^+_∞. Moreover, both u^+_∞ and u^-_∞ extend to origin and $u^+_\infty(0) = u^-_\infty(0)$.

Let us turn to the Gromov compactness theorem for curves with boundary on totally real submanifolds. To give a precise statement we need to modify the definition of the Gromov convergence (*Definition 1.6*). The reason to do it is the following. Considering *open* curves C_n with changing complex structures, we want to fix some kind of a common "neighborhood of infinity" $i_n : C^* \hookrightarrow C_n$ of every C_n. Thus we can imagine that all changes of complex structure take place "outside of infinity", *i.e.* in relatively compact part $C_n \backslash i_n(C^*) \Subset C_n$. This is done to insure that C_n do not approach to infinity in an appropriate moduli space.

On the other hand, it is more natural to consider curves $(\overline{C}_n, u_n)$ with totally real boundary conditions as compact objects without "infinity". In fact, in this case the behavior of u_n near the boundary ∂C_n can be controlled. The obtained apriori estimates near "totally real boundary" can be viewed as a part of such a "control". So for curves with totally real boundary conditions we can hope to extend the Gromov convergence up to boundary.

Further, as in the "inner case", an appropriate modification of the Gromov convergence in this case should allow boundary bubbling and appearance of boundary nodes. This means, however, that the structure of the boundary can change during approach to the limit curve and cannot be considered as fixed. Instead of it one should fix a type of boundary conditions. We shall consider the following general situation.

Let $u_n : \overline{C}_n \to X$ be a sequence of stable J_n-holomorphic over X with parametrizations $\delta_n : \overline{\Sigma} \to \overline{C}_n$. Let also $\boldsymbol{\beta} = \{\beta_i\}_{i=1}^m$ be a collection of arcs β_i in $\partial\Sigma$, $\{W_i\}_{i=1}^m$ a collection of real n-dimensional manifolds, $f_{n,i} : W_i \to X$ a sequence of totally real immersions and $u^b_{n,i} : \beta_{n,i} \to W_i$ a sequence of continuous maps from $\beta_{n,i} := \delta_n(\beta_i)$. Assume that $\cup_{i=1}^m \beta_i = \partial\Sigma$ and that the interiors of β_i are mutually disjoint and do not intersect the pre-images of boundary nodal points of C_n. Then $\boldsymbol{W}_n := \{(W_i, f_{n,i})\}_{i=1}^m$ are totally real boundary conditions on $(\overline{C}_n, u_n)$ of the same type $\boldsymbol{\beta}$.

Definition 5.6. *In the situation above we say that the sequence of boundary conditions $\boldsymbol{W}_n$ of the same type $\boldsymbol{\beta}$ converges h-uniformly transversally to J^*-totally*

real boundary conditions $\boldsymbol{W}$ on subsets $A_i \subset W_i$ if

i) $\boldsymbol{W} = \{(W_i, f_i)\}_{i=1}^m$ where $f_i : W_i \to X$ are J^*-totally real immersions;

ii) $f_{n,i}$ converge to f_i in C^1-topology and this convergence is h-uniform on A_i;

iii) for any n immersions $\{f_{n,i}\}_{i=1}^m$ are mutually h-uniformly transversal along A_i with parameters $\delta > 0$ and M, and this parameters are independent of n.

Note that the condition *iii)* implies that the limit immersions f_i are also mutually h-uniformly transversal along A_i with the same parameters $\delta > 0$ and M.

Definition 5.7. *We say that the sequence $(\overline{C}_n, u_n)$ converges up to boundary to a stable J^*-holomorphic curve $(\overline{C}_\infty, u_\infty)$ over X if the parametrizations $\sigma_n : \overline{\Sigma} \to \overline{C}_n$ and $\sigma_\infty : \overline{\Sigma} \to \overline{C}_\infty$ can be chosen in such a way that the following holds:*

i) $u_n \circ \sigma_n$ converges to $u_\infty \circ \sigma_\infty$ in $C^0(\overline{\Sigma}, X)$-topology;

ii) if $\{a_k\}$ is the set of the nodes of C_∞ and $\{\gamma_k\}$, $\gamma_k := \sigma_\infty^{-1}(a_k)$ are the corresponding circles and arcs in $\overline{\Sigma}$, then on any compact subset $K \Subset \overline{\Sigma} \backslash \cup_k \gamma_k$ the convergence $u_n \circ \sigma_n \to u_\infty \circ \sigma_\infty$ is $L^{1,p}(K, X)$ for all $p < \infty$;

iii) for any compact subset $K \Subset \overline{\Sigma} \backslash \cup_k \gamma_k$ there exists $n_0 = n_0(K)$ such that $\sigma_n^{-1}(\{a_k\}) \cap K = \varnothing$ for all $n \geqslant n_0$ and complex structures $\sigma_n^* j_{C_n}$ smoothly converge to $\sigma_\infty^* j_{C_\infty}$ on K.

Theorem 5.9. *Fix a metric h on X, an h-complete subset $A \subset X$, and subsets $A_i \subset W_i$. Suppose that:*

a) J_n are continuous almost complex structures on X, converging h-uniformly on A to a continuous almost complex structure J^*;

b) $u_n(C_n) \subset A$ and $\mathrm{area}\,[u_n(C_n)] \leqslant M$ with a constant M independent of n;

c) $\boldsymbol{W}_n := \{(W_i, f_{n,i})\}_{i=1}^m$ are totally real boundary conditions of the same type $\boldsymbol{\beta} = \{\beta_i\}_{i=1}^m$, such that $\boldsymbol{W}_n$ converge h-uniformly transversally to a boundary condition $\boldsymbol{W} = \{(W_i, f_i)\}_{i=1}^m$ on subsets $A_i \subset W_i$;

d) immersions $f_i : W_i \to (X, J^*)$ are h-uniformly totally real along A_i;

e) there exist maps $u_{i,n}^b : \beta_i \to A_i \subset W_i$, realizing boundary conditions $\boldsymbol{W}_n$.

Then there exits a subsequence of $\{(\overline{C}_n, u_n)\}$, still denoted $\{(\overline{C}_n, u_n)\}$, and parametrisations $\sigma_n : \overline{\Sigma} \to \overline{C}_n$, such that (C_n, u_n, σ_n) converges up to boundary to a stable J^*-holomorphic curve $(\overline{C}_\infty, u_\infty, \sigma_\infty)$ over X.

If, in addition, $A_i \subset W_i$ are $f_i^* h$-complete, then the limit curve $(\overline{C}_\infty, u_\infty)$ satisfies real boundary conditions $\boldsymbol{W}$ with maps $u_i^b : \beta_i \to A_i \subset W_i$.

Our main idea of the proof is to apply arguments used in the proof of *Theorem 1.1.* To do so we replace every pair (C_n, u_n) by a triple (C_n^d, τ_n, u_n^d) where C_n^d is the *Schottky double* of C_n with an antiholomorphic involution τ_n and $u_n^d : C_N^d \to X$ a τ_n-invariant map. Then we shall change all the constructions of the proof to make them τ_n-invariant in an appropriate sense. In particular, the convergence $(C_n^d, \tau_n, u_n^d) \longrightarrow (C_\infty^d, \tau_\infty, u_\infty^d)$ will be equivalent to the convergence $(C_n, u_n) \longrightarrow (C_\infty, u_\infty)$.

We start with construction of the Schottky double of a nodal curve $\overline{C}$ with boundary. Take two copies $\overline{C}^+ \equiv \overline{C}$ and $\overline{C}^-$ of $\overline{C}$. Equip $\overline{C}^-$ with the opposite complex structure, so that the identity map $\tau : \overline{C}^+ \to \overline{C}^-$ becomes now antiholomorphic. Glue $\overline{C}^+$ and $\overline{C}^-$ together along their boundaries identifying ∂C^+ and ∂C^- by means of the identity map $\tau : \partial C^+ \xrightarrow{\cong} \partial C^-$. The union $C^d := \overline{C}^+ \cup_{\partial C} \overline{C}^-$

possesses the unique structure of a closed nodal curve, which is compatible with imbeddings $\overline{C}^{\pm} \hookrightarrow C^d$. The boundary ∂C becomes the fixed point set of τ.

The map τ induces an antiholomorphic involution of C^d which we also denote by τ. We call the obtained curve C^d the *Schottky double* of $\overline{C}$. Note that every boundary nodal point $a_i \in \partial C$ defines a τ-invariant nodal point a_i on C^d, whereas an inner nodal point $b_i \in C$ defines a pair of nodal points $b_i^{\pm}$ on C^d interchanged by τ. If $\sigma : \overline{\Sigma} \to \overline{C}$ is a parametrization of $\overline{C}$, then we obtain in an obvious way the double Σ^d with the involution $\tau : \Sigma^d \to \Sigma^d$ and the parametrization $\sigma^d : \Sigma^d \to C^d$ compatible with the involutions.

Remark. The introduced notation C^d for the *Schottky double* of a nodal curve $\overline{C}$ with boundary coincides with the one for the *holomorphic double*, used in *Section 2*. Since in this section only the Schottky double is considered, this should not lead to confusion.

Suppose additionally that an almost complex structure J on X and a J-holomorphic map $u : \overline{C} \to X$ are given. Suppose also that the curve $(\overline{C}, u)$ satisfies the totally real boundary conditions $\boldsymbol{W}$ of type $\boldsymbol{\beta}$. In particular, $\boldsymbol{\beta}$ defines the certain system of arcs $\{\beta_i\}$ on ∂C. In order to take into account the type of boundary conditions, we fix the ends of β_i which are not boundary nodal points of $\overline{C}$, and declare these points to be marked points of C^d. Note that these ones and the nodal points are the only "corner" points of $(\overline{C}, u)$. The latter means, that in a neighborhood of these points the map u can be $L^{1,p}$-smooth not for all $p < \infty$. The example in *Remark* after *Corollary 5.7* explains the notion "corner point". Considering the Schottky double, we shall always equip C^d with this set of marking points. Note also that every boundary circle of $\overline{C}$ contains at least one nodal or marked point as above.

For $(\overline{C}, u)$ as above, we extend the J-holomorphic map $u : \overline{C} \to X$ to a map $u^d : C^d \to X$ by setting $u^d(x) := u(\tau(x))$ for $x \in C^-$. By the construction, u^d is τ-invariant, $u^d \circ \tau = u^d$, but u^d *is not J-holomorphic* (with the only trivial exception $u \equiv const$). However, the analysis already done in this section provides necessary $L^{1,p}$-estimates for u^d, at least for some $p^* > 2$.

In the situation of *Theorem 5.9*, such an exponent $p^* > 2$ can be chosen the same for all curves $(\overline{C}_n, u_n)$, it depends only on the topology of $\overline{C}_n$ and the geometry of immersions $f_n : W_n \to X$. In particular, every u_n^d is continuous.

Next step of the proof is to find a τ_n-invariant decomposition of C_n^d into pants. This implies that the corresponding graph Γ_n becomes τ_n-invariant. In the construction which follows we shall use the fact that τ_n is an isometry on the union of the non-exceptional components of C_n^d. This is provided by uniqueness of the intrinsic metric.

Lemma 5.10. *Let C be a nodal curve with boundary, $\sigma : \overline{\Sigma} \to \overline{C}$ a parametrization, and $\{x_i\}_{i=1}^m$ a set of marked points on boundary ∂C. Let C^d be the Schottky double of C with the anti-holomorphic involution τ.*

Then there exists a τ-invariant decomposition of $C^d \backslash \{marked\ points\}$ into pants, such that the intrinsic length of corresponding boundary circles is bounded by a constant l^+ depending only on genus g of Σ^d and the number of marked points m.

Moreover, every short geodesic appears as a boundary circle of some pants of the decomposition.

Remark. Recall (see *Remark* on page 25) that a closed geodesic γ is called *short* if $\ell(\gamma) < l^*$, where l^* is the universal constant l^* with the following property: For any simple closed geodesics γ' and γ'' on the conditions $\ell(\gamma') < l^*$ and $\ell(\gamma'') < l^*$ imply $\gamma' \cap \gamma'' = \varnothing$.

Proof. Since genus of the parameterizing real surface Σ^d and the number of marked points is fixed, we obtain a uniform upper bound on possible genus and the number of marked points of non-exceptional components of C^d, as also on the number of exceptional components. This implies that there exists a decomposition of every non-exceptional component C_i of C^d into pants S_α, such that the intrinsic length of boundary circles of S_α is bounded by the constant l^+ depending only on g and m. The idea of the proof of our lemma is to show that the construction of such a decomposition, given in [Ab], Ch.II, § 3.3, can be modified to produce a τ-invariant decomposition.

Let us first describe the construction itself, say, for a given smooth curve C^* with marked points $\{x_i\}$ of non-exceptional type. The procedure is done inductively by choosing at every step a non-trivial simple closed geodesic $\gamma_{J^*} \subset C^*\backslash\{$*marked points*$\}$, disjoint from already chosen geodesic γ_j, $j < J^*$. Moreover, at every step there exists a geodesic γ_{J^*} as above whose intrinsic length is bounded by a constant $l^+_{J^*}$ depending only on genus of C^*, the number of marked points, and the maximum of the lengths of the already chosen geodesics γ_j, $j < J^*$.

Take any non-exceptional component C^d_i of C^d. Two cases can happen: either C^d_i is τ-invariant, or $\tau\big(C^d_i\big)$ is another component $C^d_{i'}$. These cases are distinguished by the property whether C^d_i intersects the boundary ∂C (first case) or not (second one).

The existence of τ-invariant decomposition into pants for every pair of non-exceptional components C^d_i and $\tau(C^d_i) \neq C^d_i$ is obvious. We choose an appropriate decomposition of C^d_i and transfer it on $\tau(C^d_i)$ by means of τ.

It remains to consider the case of a τ-invariant non-exceptional component C^d_i.

Suppose that on some step we have already chosen a τ-invariant set $\{\gamma_1, \ldots, \gamma_{J^*-1}\}$ of simple disjoint geodesics on $C^d_i\backslash\{$*marked points*$\}$. Take a simple geodesic γ of the length $\ell(\gamma) \leqslant l^+_{J^*}$, where $l^+_{J^*}$ is the upper bound introduced above. By the construction of the double C^d, the fixed point set of τ on C^d_i is $C^d_i \cap \partial C$ and is non-empty. Denote $C_i := C \cap C^d_i$, so that $C^d_i \cap \partial C = \partial C_i$. Note that any boundary circle of C_i contains at least one marked point of C^d_i. Consequently, it has an infinite length w.r.t. the intrinsic metric on $C^d_i\backslash\{$*marked points*$\}$. Thus the chosen geodesic γ can not lie on ∂C_i. Only 3 cases can happen.

Case 1. γ is disjoint from ∂C_i. Then γ lies either in C_i or in $\tau(C_i)$. In any case, $\gamma \cap \tau(\gamma) = \varnothing$. Thus we can set $\gamma_{J^*} = \gamma$ and $\gamma_{J^*+1} = \tau(\gamma)$, getting the τ-invariant set $\{\gamma_1, \ldots, \gamma_{J^*+1}\}$ of simple disjoint geodesic. This will be the next 2 steps of our construction.

Case 2. $\gamma \cap \partial C_i \neq 0$ and γ is τ-invariant. We set $\gamma_{J^*} = \gamma$ and proceed inductively further. Note that in this case $\gamma \cap \partial C_i$ consists of 2 points, in which γ is orthogonal to ∂C_i.

Case 3. This time $\gamma \cap \partial C_i \neq 0$, but $\gamma \neq \tau(\gamma)$. Define arcs $\gamma^+ := \gamma \cap \overline{C}_i$ and $\gamma^- := \gamma \cap \tau(\overline{C}_i)$, the parts of γ inside and outside of C_i (see Fig. 7). Consider the following free homotopy classes of closed circles on C_i^d:

1) $[\tilde\gamma_1] := [\gamma^+ \cup \tau(\gamma^-)]$;
2) $[\tilde\gamma_2] := [\gamma^+ \cup \tau(\gamma^-)]$;
3) $[\tilde\gamma_3] := [\gamma^- \cup \tau(\gamma^-)]$;
4) $[\tilde\gamma_4] := [\gamma^+ \cup \tau(\gamma^-) \cup \gamma^- \cup \tau(\gamma^+)]$.

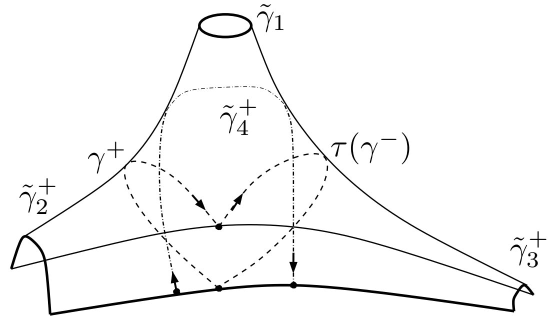

Fig. 7. Geodesics on C_i.

The last expression means that we move along corresponding arcs in the prescribed order, as it is shown on Fig. 7. Note that only one part of C_i^d is drawn, namely C_i. The rest of the picture is symmetric w.r.t. the involution τ. Thus we can see only the half of geodesics in classes $[\tilde\gamma_i]$, $i = 2,3,4$.

Each of classes $[\tilde\gamma_k]$ either is represented by a closed geodesic or corresponds to a wind around some marked point of C_i^d. To shorten notations, we say in the last case that the class $[\tilde\gamma_i]$ corresponds to a marked point of C_i^d.

If one of the classes $[\tilde\gamma_k]$, $k = 1,2,3$, is represented by the geodesic $\tilde\gamma_k$, which is different and disjoint from the already chosen geodesics γ_j, $j < J^*$, then we can set $\gamma_{J^*} = \tilde\gamma_k$. If $k = 1$ we set also $\gamma_{J^*} = \tilde\gamma_1$ and $\gamma_{J^*+1} = \tau(\tilde\gamma_1)$. Then we proceed inductively further.

To finish the proof it remains to consider the following situation: Under conditions of *Case 3*, each of the classes $[\tilde\gamma_k]$, $k = 1,2,3$, either corresponds to a marked point, or is represented by a closed geodesic $\tilde\gamma_k$, which intersects or coincides with one from the already chosen geodesics γ_j, $j < J^*$.

We claim that a proper intersection can not happen, *i.e.* each class $[\tilde\gamma_k]$, $k = 1,2,3$, either corresponds to a marked point, or is represented by an already chosen geodesic γ_j, $j < J^*$. To show this we note that $\gamma_j \cap \tau(\gamma) = \varnothing$ for all $j < J^*$. Otherwise we could have a contradiction with the conditions $\gamma_j \cap \gamma = \varnothing$ and τ-invariance of the set of the geodesics γ_j, $j < J^*$. Consequently, each class $[\tilde\gamma_k]$ is represented by a circle $\alpha_k \subset C_i^d \backslash \{\textit{marked points}\}$, $k = 1,2,3,4$, with $\alpha_k \cap \gamma_j = \varnothing$.

Now assume that the proper intersection of $\tilde\gamma_k$ and some γ_j, $j < J^*$, does have place. Let $\ell_k := \ell(\tilde\gamma_k)$ be the intrinsic metric of $\tilde\gamma_k$. As in the proof of *Lemma 2.2* construct the annulus $A = \{(\rho,\theta) : |\rho| < \frac{\pi^2}{\ell}\} \times \{0 \leqslant \theta \leqslant 2\pi\}$ with the metric $(\frac{\ell_k}{2\pi}/\cos\frac{\ell_k\rho}{2\pi})^2(d\rho^2 + d\theta^2)$ and an isometric covering of $C_i^d \backslash \{\textit{marked points}\}$ by A, which sends the geodesic $\beta_k := \{\rho = 0\} \subset A$ onto $\tilde\gamma_k \subset C_i^d$. Find a lift of γ_j to a geodesic line $L_j \subset A$ with $L_j \cap \beta_k \neq \varnothing$, and a lift of the circle α_k to a circle $\tilde\alpha_k \subset A$ homotopic to β_k. Then the intersection $L_j \cap \beta_k$ must consist of exactly one point, and consequently, the homology intersection index $[L_j] \cdot [\beta_k]$ is equal to ± 1. This would imply that $[L_j] \cdot [\tilde\alpha_k] = [L_j] \cdot [\beta_k] \neq 0$ and consequently $L_j \cap \tilde\alpha_k \neq \varnothing$. But this would contradict to $\gamma_j \cap \alpha_k = \varnothing$.

Summing up, we see that in our situation we must have a picture of *Fig. 7*. Namely, the both geodesics γ and $\tau(\gamma)$ lie in a τ-invariant domain Ω on C_i^d with 4 components of the boundary. These components of $\partial\Omega$ are either marked points or geodesics corresponding to the classes $[\tilde\gamma_1]$, $[\tau(\tilde\gamma_1)]$, $[\tilde\gamma_2]$, $[\tilde\gamma_3]$. Finally, every boundary circle of Ω is one of the geodesics γ_j. We conclude, that the class $[\tilde\gamma_4]$ is

represented by a τ-invariant geodesic $\tilde\gamma_4$. This geodesic can be chosen at this step of construction of τ-invariant decomposition of C^d_i into pants.

Note that by construction for the intrinsic length of γ_{J^*} we get $\ell(\gamma_{J^*}) \leqslant 2\ell(\gamma) \leqslant 2l^+_{J^*}$. This means that in our construction we do not lose control of the intrinsic length of chosen geodesics. This provides the existence of a constant l^+ stated in the lemma.

Finally, the definition of a *short* geodesic provides that the geodesic γ in *Case 3* above cannot be short. This implies that the set of short geodesic on C^d is disjoint. Since the involution τ is an isometry, the set of short geodesic on C^d is also τ-invariant. Thus in our construction of decomposition into pants we can start with this set of geodesics. This shows the last statement of the lemma. □

Remark. To explain the meaning of *Lemma 5.10*, let us consider pants S with a complex structure J_S and an anti-holomorphic involution τ acting on S. It is easy to see that only two types of such an action, illustrated by Figs. 8 a) and 8 b), are possible.

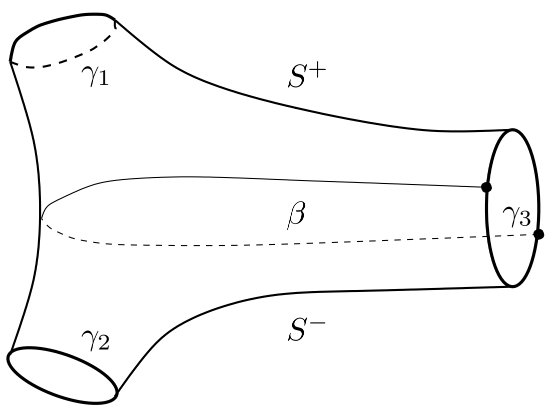

Fig. 8 a)

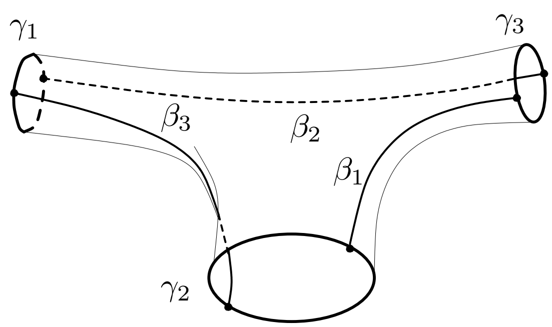

Fig. 8 b)

In the first case, Fig. 8 a), the involution τ interchanges two boundary components γ_1 and γ_2 of S and leaves the third one γ_3 invariant. The fixed point set β of τ is a geodesic arc with both ends on the τ-invariant boundary component γ_3. This case includes subcases when some boundary components of S are not geodesics but marked points (*i.e.* punctures). In particular, if γ_3 is a marked point, then the set β is an (infinite) geodesic line with both ends approaching to γ_3. The set β divides S into two parts, S^+ and S^- (see Fig. 8 a)), which are interchanged by τ. Topologically, each part $S^\pm$ is an annulus.

In the second case, Fig. 8 b), all three boundary components γ_1, γ_2, and γ_3 are invariant. The fixed point set of τ consists of geodesic arcs β_1, β_2, and β_3. These are the shortest simple geodesics between γ_2 and γ_3, resp. γ_3 and γ_1, and resp. γ_2 and γ_3. If some boundary component of S is not a geodesic but a marked point, then corresponding arcs have ends of infinite length approaching to this boundary component. The arcs β_k, $k = 1,2,3$, divide S into two parts, S^+ and S^- (see Fig. 8 b)), which are interchanged by τ. In this case, each part $S^\pm$ is topologically a disc.

We call pieces $S^\pm$ *half-pants of first or second type* respectively. Note that in both cases τ-invariant arcs β or β_i are orthogonal to corresponding boundary circles γ_j.

Return to the situation of a nodal curve $\overline{C}$ with boundary and marked points. Let C^d be the Schottky double and τ the anti-holomorphic involution. Suppose that $C^d\backslash\{\textit{marked points}\}$ is non-exceptional. Use *Lemma 5.10* and find a τ-invariant decomposition into pants $C^d = \cup_j S_j$. Set $S_j^+ := S_j \cap \overline{C}$. Then we obtain a decomposition $\overline{C} = \cup_j S_j^+$, such that the pieces S_j^+ are either pants (which means $S_j^+ = S_j$), or half-pants of the first or the second type. This decomposition is a suitable one for the situation of the Gromov convergence up to boundary of curves with totally real boundary conditions. In particular, we obtain arcs $\beta_{j,k}$ as τ-fixed point sets of S_j^+, which define a decomposition $\partial C = \cup_{j,k}\beta_{j,k}$ of the boundary of $\overline{C}$. The collection $\boldsymbol{\beta}' := \{\beta_{j,k}\}$ of these arcs satisfies the condition *i)* of *Definition 5.4*, but it can be different from the collection $\boldsymbol{\beta} = \{\beta_i\}$ which was given. The reason is that in construction of the pants-decomposition $C^d = \cup_j S_j$ we can subdivide original arcs $\beta_i \in \boldsymbol{\beta}$ into smaller pieces, so that every arc $\beta_i \in \boldsymbol{\beta}$ is a union of arcs $\beta_{j,k}$ from $\boldsymbol{\beta}'$. This means compatibility of $\boldsymbol{\beta}$ and $\boldsymbol{\beta}'$.

The next step is to establish a generalization of *Theorem 4.2*. Assume that the hypothesis of *Theorem 5.9* are fulfilled. For each curve C_n denote by C_n^d its Schottky double and by τ_n the corresponding involution.

Lemma 5.11. *In the situation above, after passing to a subsequence, there exist parametrizations $\sigma_n^d : \Sigma^d \to C_n^d$, a finite covering $\mathcal{V}$ of Σ^d by open sets $\{V_\alpha\}$, and a set $\{x_1^*, \dots, x_m^*\}$ of marked points on Σ, such that the conditions* (a), (c)–(f) *of Theorem 4.2 and the following additional conditions* (b') *and* (h) *are satisfied:*

(b') $\sigma_n\{x_1^*, \dots, x_m^*\}$ is the set of marked points on C_n^d corresponding decomposition of the boundary ∂C_n into arcs $\beta_{n,i}$; moreover, each such point x_j^* lies in a single piece of covering V_α which is a disc;

(h) there exist an involution $\tau : \Sigma^d \to \Sigma^d$ which is compatible with the covering $\mathcal{V}$ and with parametrizations σ_n^d, i.e. $\mathcal{V}$ is τ-invariant and $\tau_n \circ \sigma_n^d = \sigma_n^d \circ \tau$. In particular, each marked point x_i^* of Σ^d is fixed by τ.

Remark. The condition *(g)* of *Theorem 4.2* is trivial in this case since C_n^d and Σ^d are closed.

Proof. One can use the proof of *Theorem 4.2* with minor modifications. Note that starting points of that proof were the intrinsic metric on non-exceptional components of nodal curves C_n there and the decomposition of C_n into pants. Now the existence of a τ-invariant decomposition of the curves C_n^d into pants is provided by *Lemma 5.10*, whereas the τ-invariance of the intrinsic metrics follows from the fact, that any (anti)holomorphic isomorphism of curves with marked points is an isometry w.r.t. the intrinsic metric. Thus the constructions of the proof of *Theorem 4.2* yield τ-invariant objects. The condition *(b')* does not bring much difficulty. $\square$

Now we are ready to finish

Proof of Theorem 5.9. As it was mentioned, our main idea is to modify the construction used in the proof of *Theorem 1.1* to make them τ-invariant. Main work is already done: we have necessary apriori estimates, the construction of a τ-invariant pants-decomposition of the double C_n^d of the curve $\overline{C}_n$, and the appropriate covering $\mathcal{V}$ of the real surface Σ^d parametrizing the doubles C_n^d.

As in the proof of *Theorem 1.1* we consider the curves $C_{\alpha,n} := \sigma_n^d(V_\alpha)$. Due to presence of the involutions τ_n the geometrical situation in now different. This involves new phenomena and needs additional considerations and constructions. In particular, the pieces $C_{\alpha,n}$ are divided into 2 groups depending on that whether they are disjoint from the boundary ∂C_n or intersect it. In the last case $C_{\alpha,n}$ is τ_n-invariant. In this case we shall use the notation $C^+_{\alpha,n} := C_{\alpha,n} \cap C_n$ for the part of $C_{\alpha,n}$ lying in C_n. Besides, we denote $V^+_\alpha := V_\alpha \cap \Sigma$. Then V_α appear as the union of domains V^+_α and $\tau(V^+_\alpha)$, interchanged by τ. Similar is true for $C_{\alpha,n}$.

To prove the theorem, we want to construct a refined covering $\widetilde{\mathcal{V}}$ of Σ and refined parametrizations $\tilde\sigma_n : \Sigma \to C_n$ such that for every $V_\alpha \in \widetilde{\mathcal{V}}$ the sequence $(C_{\alpha,n}, u_{\alpha,n})$ with $C_{\alpha,n} := \tilde\sigma_n(V_\alpha)$ one the following convergence types holds:

A′) *$C_{\alpha,n}$ are annuli of infinitely growing conformal radii l_n disjoint from ∂C_n, and the conclusions of Lemma 3.7 hold;*

A″) *$C_{\alpha,n}$ are τ_n-invariant annuli of infinitely growing conformal radii l_n and the conclusions of Lemma 5.8 are valid for $\Theta(0, l_n) \cong C^+_{\alpha,n} := C_{\alpha,n} \cap C_n$;*

B′) *every $C_{\alpha,n}$ is disjoint from ∂C_n and isomorphic to the standard node $\mathcal{A}_0 = \Delta \cup_{\{0\}} \Delta$, such that the compositions $V_\alpha \xrightarrow{\sigma_{\alpha,n}} C_{\alpha,n} \xrightarrow{\cong} \mathcal{A}_0$ define the same parametrisations of $\mathcal{A}_0$ for all n; furthermore, the induced maps $\tilde u_{\alpha,n} : \mathcal{A}_0 \to X$ strongly converge;*

B″) *every $C_{\alpha,n}$ is τ_n-invariant and $C^+_{\alpha,n} := C_{\alpha,n} \cap C_n$ is isomorphic to the standard boundary node $\mathcal{A}^+_0 = \Delta^+ \cup_{\{0\}} \Delta^+$, such that for $V^+_\alpha := V_\alpha \cap \Sigma$ the compositions $V^+_\alpha \xrightarrow{\sigma^+_{\alpha,n}} C^+_{\alpha,n} \xrightarrow{\cong} \mathcal{A}^+_0$ define the same parametrisations of $\mathcal{A}^+_0$ for all n; furthermore, the induced maps $\tilde u^+_{\alpha,n} : \mathcal{A}^+_0 \to X$ strongly converge;*

C) *the structures $\sigma_n^* j_n\big|_{V_\alpha}$ and the maps $u_{\alpha,n} \circ \sigma_{\alpha,n} : V_\alpha \to X$ strongly converge.*

In the case *B″)* the strong convergence of maps $\tilde u^+_n : \mathcal{A}^+_0 \to X$ is the one in the L^{1,p^*}-topology for some $p^* > 2$ *up to the boundary intervals containing the nodal point*. An equivalent requirement is the usual L^{1,p^*}-convergence of the doubles $\tilde u^d_n : C_{\alpha,n} \to X$ on compact subsets of $C_{\alpha,n} \cong \mathcal{A}_0$.

To obtain a desired refinement we use the same inductive procedure as in the proof of *Theorem 1.1*. To insure convergence near boundary ∂C_n, we take a new value for the constant determining the inductive step. We choose a positive ε^b, such that $\varepsilon^b \leqslant \varepsilon$ and such that all apriori estimates of this section are valid for maps with area $\leqslant 3\varepsilon^b$. This will yield the convergence of type *A)–C)* for sequences of curves with totally real boundary conditions and with the upper bound ε^b on area.

In fact, essential modifications of constructions of *Theorem 1.1* are needed only if the covering piece V_α is τ-invariant. Indeed, if V_α is not τ-invariant, then we can apply all the argumentations and constructions used in *Cases 1)–4)* in the proof of *Theorem 1.1*, and then "transfer" them onto $\tau(V_\alpha)$ by means of τ. This gives the inductive step preserving τ-invariance.

Hence, it remains consider the situation when the covering piece V_α is τ-invariant. As in *Theorem 1.1*, we must consider 4 cases:

Case 1_b): $C_{\alpha,n}$ have constant complex structure, different from the one of the standard node;

Case 2_b): $C_{\alpha,n}$ are annuli of changing conformal radii R_n, such that $R_n \to R < \infty$

Case 3$_b$): $C_{\alpha,n}$ are isomorphic to the standard node, so that $C^+_{\alpha,n}$ are isomorphic to the standard boundary node $\mathcal{A}^+_0$;

Case 4$_b$): $C_{\alpha,n}$ are annuli of infinitely growing conformal radii R_n.

The subindex $(\cdot)_b$ indicates that we consider the cases when V_α intersects the *b*oundary of Σ. As it was mentioned the last property is equivalent to the τ-invariantness of V_α. References to *Cases 1)–4) without* the subindex will mean the corresponding parts of the proof of *Theorem 1.1*.

Case 1$_b$). Without loss of generality we may assume that V_α is a domain with a fixed complex structure and a fixed antiholomorphic involution τ, and that $u_{\alpha,n} : V_\alpha \to X$ is a sequence of τ-invariant maps which are (anti)holomorphic outside the set of τ-invariant points of V_α. If we have the convergence of type *C)* there is nothing to do. Otherwise we fix a τ-invariant metric on V_α compatible with the complex structure. Repeating the constructions from *Case 1)* we distinguish the "bubbling" points $y^*_1, \dots, y^*_l$ where the strong convergence fails.

Take the first such point y^*_1. Suppose y^*_1 is disjoint from $\partial\Sigma$, Then we may assume that $y^*_1 \in V^+_\alpha$. Thus we can repeat the rest of the constructions from *Case 1)*. The only correction needed at this place is that the neighborhood $\Delta(y^*_1, \varrho)$ of y^*_1 must be small enough and lie in V^+_α. Transfering all these constructions into $\tau(V_\alpha)$, we realize the inductive step preserving τ-invariance.

It remains to consider the case when $y^*_1 \in \partial\Sigma$. This means that y^*_1 is τ-invariant. Let z be a holomorphic coordinate in a neigborhood of y^*_1 on V_α, such that $z = 0$ in y^*_1, the involution τ corresponds to the conjugation $z \mapsto \bar z$, and $\mathsf{Im}\, z > 0$ in Σ. Find the sequences $r_n \longrightarrow 0$ of radii and $x_n \to y^*_1$ using the constructions from *Case 1)*. Note that the sequence $\tau(x_n)$ have the same property. Thus, replacing some points x_n by $\tau(x_n)$, we may additionally assume that all x_n lie in $\overline{V}^+_\alpha$. Let $v_n : \Delta(0, \frac{\varrho}{2r_n}) \to (X, J_n)$ be the rescalings of maps u_n defined by $v_n(z) := u_n(x_n + \frac{z}{r_n})$. Argumentations of *Case 1* shows that there exists the limit $v_\infty : \mathbb{C} \to X$ of (a subsequence of) $\{v_n\}$ which extends to a map $v_\infty : S^2 \to X$.

Denote by ρ_n the distance from x_n to $\partial\Sigma$ and by $\tilde x_n$ the point on $\partial\Sigma$ closest to x_n. Then $x_n = \tilde x_n + i\rho_n$ in the coordinate z introduced above. Besides, $\mathsf{lim}\, \tilde x_n = y^*_1$. We consider 2 subcases according to possible behavior of ρ_n and r_n.

Subcase 1$'_b$): $\{\frac{\rho_n}{r_n}\}$ is bounded. Passing to a subsequence, we may assume that $\frac{\rho_n}{r_n}$ converges. Fix an upper bound b for the sequence $\frac{\rho_n}{r_n}$. In particular, $b \geqslant \mathsf{lim}\, \frac{\rho_n}{r_n}$.

For $n >> 1$ define maps $v_n : \Delta(0, \frac{\varrho}{2r_n} - b) \to (X, J_n)$ and $\tilde v_n : \Delta(0, \frac{\varrho}{2r_n} - b) \to (X, J_n)$ setting $v_n(z) := u_n(x_n + r_n z)$ and $\tilde v_n(z) := u_n(\tilde x_n + r_n z)$ respectively. Then every $\tilde v_n$ is the shift of the map v_n by $i\frac{\rho_n}{r_n}$, *i.e.* $\tilde v_n(z) = v_n\big(z + i\frac{\rho_n}{r_n}\big)$. The arguments of *Case 1)* show that v_n converge on compact subsets of $\mathbb{C}$ to a non-constant map. Consequently, $\tilde v_n$ also converge on compact subsets of $\mathbb{C}$ to a non-constant map $\tilde v_\infty : \mathbb{C} \to X$. Moreover, since $\mathsf{area}\,(\tilde v_\infty(\mathbb{C}))$ is finite, $\tilde v_\infty$ extends to a map $\tilde v_\infty : S^2 \to X$. By the choice of ε^b, $\mathsf{area}\,(\tilde v_\infty(S^2)) \geqslant 3\varepsilon^b$. Changing the choice of the constant b, we can additionally assume that $\mathsf{area}\,(\tilde v_\infty(\Delta(0, b)) \geqslant 2\varepsilon^b$. Then for all sufficiently big n we get

$$\mathsf{area}\,(\tilde v_n(\Delta(0, b)) \geqslant \varepsilon^b \tag{5.?10}$$

For $n >> 1$ we define the coverings of V_α by 3 sets

$$V^{(n)}_{\alpha,1} := V_\alpha \backslash \overline{\Delta}(0, \tfrac{\varrho}{2}), \qquad V^{(n)}_{\alpha,2} := \Delta(0, \varrho) \backslash \overline{\Delta}(\tilde x_n, br_n), \qquad V^{(n)}_{\alpha,3} := \Delta(\tilde x_n, 2br_n).$$

Fix n_0 sufficiently big. Denote $V_{\alpha,1} := V^{(n_0)}_{\alpha,1}$, $V_{\alpha,2} := V^{(n_0)}_{\alpha,2}$, and $V_{\alpha,3} := V^{(n_0)}_{\alpha,3}$. There exist diffeomorphisms $\psi_n : V_1 \to V_1$ such that $\psi_n : V_{\alpha,1} \to V^{(n)}_{\alpha,1}$ is identity, $\psi_n : V_{\alpha,2} \to V^{(n)}_{\alpha,2}$ is a diffeomorphism, and $\psi_n : V_{\alpha,3} \to V^{(n)}_{\alpha,3}$ is biholomorphic w.r.t. the complex structures, induced from C_n by means of σ^d_n. Note that the sets $V^{(n)}_{\alpha,i}$ are τ-invariant. Moreover, we can choose the maps ψ_n in such a way that ψ_n are also τ-invariant.

The covering $\{V_{\alpha,1}, V_{\alpha,2}, V_{\alpha,3}\}$ of V_1 and parametrizations $\tilde\sigma_n := \sigma_{\alpha,n}\circ\psi_n : V_1 \to C_{\alpha,n}$ satisfy the conditions of *Lemma 5.11*. Moreover, inequality (5.?1) implies $\mathsf{area}\,(u_n(\tilde\sigma_n(V_{\alpha,i}))) \leqslant (N-1)\varepsilon^b$. Consequently, we can apply the inductive assumptions for the sequence of curves $\tilde\sigma_n(V_{\alpha,i})$ and finish the proof by induction.

Subcase $1''_b$): $\{\frac{\rho_n}{r_n}\}$ is unbounded. Passing to a subsequence, we may assume that $\frac{\rho_n}{r_n}$ increases infinitely. However, $\rho_n \longrightarrow 0$ since $x_n \longrightarrow y^*_1 \in \partial\Sigma$.

Define maps $v_n : \Delta(0, \frac{\varrho}{2r_n}) \to (X, J_n)$ setting $v_n(z) := u_n(x_n + r_n z)$. As in *Case 1)*, v_n converge on compact subsets of $\mathbb{C}$ to a non-constant map $v_\infty : \mathbb{C} \to X$, which extends to a map from the whole sphere S^2. Choose $b > 0$ satisfying (5.?1).

For $n >> 1$ we define the coverings of V_α by 5 sets

$$V^{(n)}_{\alpha,1} := V_\alpha \backslash \overline\Delta(0, \tfrac{\varrho}{2}), \quad V^{(n)}_{\alpha,2} := \Delta(0,\varrho)\backslash\overline\Delta(0, 2\rho_n)$$
$$V^{(n)}_{\alpha,3} := \Delta(0, 4\rho_n)\backslash\big(\overline\Delta(x_n, br_n) \cap \overline\Delta(\tau(x_n), br_n)\big)$$
$$V^{(n)}_{\alpha,4} := \Delta(x_n, 2br_n), \quad V^{(n)}_{\alpha,5} := \Delta(\tau(x_n), 2br_n).$$

Fix n_0 sufficiently big. Denote $V_{\alpha,i} := V^{(n_0)}_{\alpha,i}$, $i = 1, \ldots, 5$. Then for every $n >> 1$ there exists a diffeomorphism $\psi_n : V_1 \to V_1$ with the following properties:

i) ψ_n maps $V_{\alpha,i}$ onto $V^{(n)}_{\alpha,i}$ diffeomorphically;

ii) $\psi_n : V^{(n)}_{\alpha,1} \to V^{(n)}_{\alpha,1}$ is the identity;

iii) $\psi_n : V_{\alpha,2} \to V^{(n)}_{\alpha,2}$ and $\psi_n : V_{\alpha,3} \to V^{(n)}_{\alpha,3}$ are diffeomorphisms;

iv) $\psi_n : V_{\alpha,3} \to V^{(n)}_{\alpha,3}$ is biholomorphic w.r.t. the complex structures, induced from C_n by means of σ^d_n; and, finally

v) ψ_n are τ-invariant: $\tau\circ\psi_n = \psi_n\circ\tau$.

Note that the last property is obtained due to the fact that the sets $V^{(n)}_{\alpha,i}$ are τ-invariant. The rest constructions are the same as in *Subcase $1'_b$)*.

Case 2_b). Consider the parametrizations $\sigma_n : V_\alpha \to C_{\alpha,n}$. Without loss of generality we may assume that the complex strictures $\sigma^*_n j_n\big|_{V_\alpha}$ are constant near boundary ∂V_α and converge to some complex structure. If we have the convergence of type *C)*, *i.e.* the strong convergence, there is nothing to do. Otherwise there exists only a finite set of points $\{y^*_1, \ldots, y^*_l\}$ where the strong convergence fails. Changing the parametrizations σ_n, we may additionally assume that the strictures $\sigma^*_n j_n\big|_{V_\alpha}$ are constant in the neighborhood of these points. Then we repeat the argumentations of *Case 1_b)*.

Case 3_b). Fix identifications $C_{\alpha,n} \cong \mathcal{A}_0$ such that every $C^+_{\alpha,n}$ is mapped onto $\mathcal{A}^+_0$ and such that the induced parametrization maps $\sigma_{\alpha,n} : V_\alpha \to \mathcal{A}_0$ are the same for all n and τ-invariant. Fix the standard representation of $\mathcal{A}_0$ as the union of two

discs Δ' and Δ'' with identification of the centers $0 \in \Delta'$ and $0 \in \Delta''$ into the nodal point of $\mathcal{A}_0$, still denoted by 0. Let $\Delta'(x, r)$ denote the subdisc of Δ' with the center x and the radius r.

Denote by $u'_n : \Delta' \to X$ and $u''_n : \Delta'' \to X$ the corresponding "components" of the maps $u_{\alpha,n} : C_{\alpha,n} \to X$. Find the common collection of bubbling points y^*_i for both sequences of maps $u'_n : \Delta' \to X$ and $u''_n : \Delta'' \to X$. If there are no bubbling points, then we obtain the convergence type *B)* and the proof can be finished by induction. Otherwise consider the first such point y^*_1, which lies, say, on Δ'. If y^*_1 is distinct from the nodal point $0 \in \Delta'$, then we simply repeat all the construction *Case 1_b)*.

It remains to consider the case $y^*_1 = 0 \in \Delta'$. The following modifications of the argumentations are needed. Repeat the construction of the radii $r_n \longrightarrow 0$ and the points $x_n \longrightarrow y^*_1 = 0$ from *Case 1_b*. Then $\{x_n\}$ is a sequence in the half-disk $\delta'^+ := \{z \in \Delta' : \mathsf{Im}\, z \geqslant 0\}$. Set $\tilde{x}_n := \mathsf{Re}\,(x_n)$, $\rho_n := \mathsf{Im}\,(x_n)$ and $R_n := |x_n|$. Thus $x_n = \tilde{x}_n + i\rho_n$, R_n is the distance from x_n to the point $0 = y^*_1 \in \Delta'$, whereas ρ_n is the distance from x_n to the interval $]-1, 1[\subset \Delta'$, the set τ-invariant points of Δ'. Thus $\rho_n \leqslant R_n$. Fix $\varrho > 0$ such that the disc $\Delta'(0, \varrho)$ contains no bubbling points $y^*_i \neq 0 \in \Delta'$.

Depending on the behavior of the sequences r_n, ρ_n and R_n, we consider the 4 subcases.

Subcase $3'_b$): The sequence $\left\{\frac{R_n}{r_n}\right\}$ is bounded. Then the sequences $\left\{\frac{\rho_n}{r_n}\right\}$ and $\left\{\frac{\tilde{x}_n}{r_n}\right\}$ are also bounded. Passing to a subsequence we may assume that the corresponding limits exist. Let b be some upper bound for the sequence $\left\{\frac{R_n}{r_n}\right\}$. Consider the maps $\tilde{v}_n : \Delta(0, \frac{\varrho}{2r_n} - b) \to X$ defined by $\tilde{v}_n(z) := u_n(\tilde{x}_n + \frac{z}{r_n})$. Then $\tilde{v}_n$ are τ-invariant, $\tilde{v}_n \circ \tau = \tilde{v}_n$, $\tilde{v}_n$ converge to a nonconstant map $\tilde{v}_\infty : \mathbb{C} \to X$ on compact subsets of $\mathbb{C}$, and $\tilde{v}_\infty$ extends to a map $\tilde{v}_\infty : S^2 \to X$.

Since $\tilde{v}_\infty$ is nonconstant, $\|d\tilde{v}_\infty\|^2_{L^2(S^2)} = \mathsf{area}\,(\tilde{v}_\infty(S^2)) \geqslant 3\varepsilon_b$. Choose $b > 0$ in such a way that

$$\|d\tilde{v}_\infty\|^2_{L^2(\Delta(0,b))} \geqslant 2\varepsilon_b \tag{5.?15}$$

and $b \geqslant 2 \lim \frac{R_n}{r_n} + 2$. Due to *Corollary 5.2* for $n >> 1$ we obtain the estimate

$$\|du'_n\|^2_{L^2(\Delta'(\tilde{x}_n, br_n))} = \|d\tilde{v}_n\|^2_{L^2(\Delta(0,b))} \geqslant \varepsilon_b. \tag{5.?16}$$

Note that $0 \in \Delta'(\tilde{x}_n, (b-1)r_n))$ for $n >> 1$ by the choice of b.

Define the coverings of $\mathcal{A}_0$ by 4 sets

$$\begin{aligned} W^{(n)}_1 &:= \Delta' \backslash \overline{\Delta}'(0, \tfrac{\varrho}{2}), & W^{(n)}_2 &:= \Delta'(0, \varrho) \backslash \overline{\Delta}'(\tilde{x}_n, br_n), \\ W^{(n)}_3 &:= \Delta'(\tilde{x}_n, 2br_n) \backslash \overline{\Delta}'(0, \tfrac{r_n}{2}), & W^{(n)}_4 &:= \Delta'(0, r_n) \cup \Delta'', \end{aligned}$$

and lift them to V_α by putting $V^{(n)}_{\alpha,i} := \sigma^{-1}_{\alpha,n}(W^{(n)}_i)$. Choose $n_0 >> 0$, such that $|x_n| < (b-1)r_n$ and the relation (5.?16) holds for all $n \geqslant n_0$. Set $V_{\alpha,i} := V^{(n_0)}_{\alpha,i}$. Fix diffeomorphisms $\psi_n : V_\alpha \to V_\alpha$ such that $\psi_n : V_{\alpha,1} \to V^{(n)}_{\alpha,1}$ is the identity map, $\psi_n : V_{\alpha,2} \to V^{(n)}_{\alpha,2}$ and $\psi_n : V_{\alpha,3} \to V^{(n)}_{\alpha,3}$ are diffeomorphisms, and $\psi_n : V_{\alpha,4} \to V^{(n)}_{\alpha,4}$ correspond to isomorphisms of nodes $W^{(n)}_4 \cong \mathcal{A}_0$. Set $\sigma'_n := \sigma_n \circ \psi_n$. The

choice above can be done in such a way that the refined covering $\{V_{\alpha,i}\}$ of V_α and parametrization maps $\sigma'_n : V_\alpha \to C_{\alpha,n}$ have the properties of *Lemma 5.11*. Relation (5.?16) implies the estimate $\mathsf{area}\,(u_n(\sigma'_n(V_{\alpha,i})) \leqslant (N-1)\,\varepsilon$. This provides the inductive conclusion for *Subcase 3$'_b$)*.

Subcase 3$''_b$): The sequence $\{\frac{R_n}{r_n}\}$ increases infinitely but $\{\frac{\rho_n}{r_n}\}$ remains bounded. Note that in this subcase we still have the relation $R_n \longrightarrow 0$, or equivalently, $x_n \longrightarrow 0$. On the other hand, $\lim \frac{\rho_n}{R_n} = 0$. This implies that for $\tilde{R}_n := |\tilde{x}_n|$ we have $\lim \frac{\tilde{R}_n}{R_n} = 1$ since $R_n^2 = \tilde{R}_n^2 + \rho_n^2$.

We proceed as follows. Define the maps $\tilde{v}_n : \Delta(0, \frac{\varrho}{2r_n} - b) \to X$ setting $\tilde{v}_n(z) := u'_n(\tilde{x}_n + \frac{z}{r_n})$. Then $\tilde{v}_n$ have the same properties as in *Subcase 3$'_b$)*. Choose $b > 0$ obeying the relation (5.?15). Then for $n >> 0$ we get the property (5.?16).

For $n >> 0$ define the coverings of $\mathcal{A}_0$ by 6 sets

$$W_1^{(n)} := \Delta' \backslash \overline{\Delta}'(0, \tfrac{\varrho}{2}), \qquad W_2^{(n)} := \Delta'(0,\varrho) \backslash \overline{\Delta}'(\tilde{x}_n, 2\tilde{R}_n),$$

$$W_3^{(n)} := \Delta'(\tilde{x}_n, 4\tilde{R}_n) \backslash \big(\overline{\Delta}'(\tilde{x}_n, \tfrac{\tilde{R}_n}{6}) \cup \overline{\Delta}'(0, \tfrac{\tilde{R}_n}{6})\big) \quad W_4^{(n)} := \Delta'(0, \tfrac{\tilde{R}_n}{3}) \cup \Delta'',$$

$$W_5^{(n)} := \Delta'(\tilde{x}_n, \tfrac{\tilde{R}_n}{3}) \backslash \overline{\Delta}'(\tilde{x}_n, b r_n), \qquad W_6^{(n)} := \Delta'(0, 2 b r_n),$$

and lift them to V_α by putting $V_{\alpha,i}^{(n)} := \sigma_{\alpha,n}^{-1}(W_i^{(n)})$. Choose $n_0 >> 0$, such that $R_{n_0} >> b r_{n_0}$, and set $V_{\alpha,i} := V_{\alpha,i}^{(n_0)}$. Choose diffeomorphisms $\psi_n : V_\alpha \to V_\alpha$ such that $\psi_n : V_{\alpha,1} \to V_{\alpha,1}^{(n)}$ is the identity map, $\psi_n : V_{\alpha,2} \to V_{\alpha,2}^{(n)}$, $\psi_n : V_{\alpha,4} \to V_{\alpha,4}^{(n)}$ and $\psi_n : V_{\alpha,5} \to V_{\alpha,5}^{(n)}$ are diffeomorphisms, and finally, $\psi_n : V_{\alpha,6} \to V_{\alpha,6}^{(n)}$ corresponds to isomorphisms of nodes $W_6^{(n)} \cong \mathcal{A}_0$. Set $\sigma'_n := \sigma_n \circ \psi_n$. Note that the choices can be done in such a way that $\{V_{\alpha,i}\}$ and parametrization maps $\sigma'_n : V_\alpha \to C_{\alpha,n}$ have the properties of *Lemma 5.11*. As above, we get the estimate $\mathsf{area}\,(u_n(\sigma'_n(V_{\alpha,i})) \leqslant (N-1)\,\varepsilon$ due to (5.?16). Thus we get the inductive conclusion for *Subcase 3$''_b$)* and can proceed further.

Subcase 3$'''_b$): The sequence $\{\frac{\rho_n}{r_n}\}$ increases infinitely, but $\{\frac{R_n}{\rho_n}\}$ remains bounded. Then $\{\frac{R_n}{r_n}\}$ also increases infinitely, but both sequences $\{R_n\}$ and $\{\rho_n\}$ converge to 0. We may also assume that $\{\frac{\rho_n}{R_n}\}$ and $\{\frac{\tilde{x}_n}{R_n}\}$ also converge. Set $a_1 := \lim \frac{\tilde{x}_n}{R_n}$, $a_2 := \lim \frac{\rho_n}{R_n}$, $a := a_1 + i a_2$, and $\overline{a} := a_1 - i a_2$. Note that $0 < a_2 \leqslant 1$ and that the involutions τ_n in $C_{\alpha,n}$ correspond to the complex conjugation $z \to \overline{z}$ in Δ'. In particular, $\overline{x}_n = \tau_n(x_n)$.

Consider maps $v_n : \Delta(0, \frac{\varrho}{2r_n}) \to X$ defined by $v_n(z) := u'_n(x_n + \frac{z}{r_n})$. Then the sequence $\{v_n\}$ converges on compact subsets to a nonconstant map which extends to the map $v_\infty : S^2 \to X$. Moreover, we can fix sufficiently big $b > 0$ such that for $n >> 0$ we get the property (5.?16).

For $n >> 0$ define the coverings of $\mathcal{A}_0$ by 8 sets

$$W_1^{(n)} := \Delta' \backslash \overline{\Delta}'(0, \tfrac{\varrho}{2}), \qquad W_2^{(n)} := \Delta'(0,\varrho) \backslash \overline{\Delta}'(0, 2R_n),$$

$$W_3^{(n)} := \Delta'(0, 4R_n) \backslash \big(\overline{\Delta}'(a\,R_n, \tfrac{a_2 R_n}{4}) \cup \overline{\Delta}'(\overline{a}\,R_n, \tfrac{a_2 R_n}{4}) \cup \overline{\Delta}'(0, \tfrac{a_2 R_n}{4})\big)$$

$$W_4^{(n)} := \Delta'(0, \tfrac{a_2 R_n}{3}) \cup \Delta''.$$

$$W_5^{(n)} := \Delta'(aR_n, \tfrac{a_2 R_n}{3}) \backslash \overline{\Delta}'(x_n, b r_n), \qquad W_6^{(n)} := \Delta'(x_n, 2 b r_n),$$

$$W_7^{(n)} := \Delta'(\overline{a}R_n, \tfrac{a_2 R_n}{3}) \backslash \overline{\Delta}'(\overline{x}_n, b r_n), \qquad W_8^{(n)} := \Delta'(\overline{x}_n, 2 b r_n),$$

and lift them to V_α by putting $V^{(n)}_{\alpha,i} := \sigma^{-1}_{\alpha,n}(W^{(n)}_i)$. Fix sufficiently big $n_0 > > 0$, and set $V_{\alpha,i} := V^{(n_0)}_{\alpha,i}$. Choose diffeomorphisms $\psi_n : V_\alpha \to V_\alpha$ mapping $V_{\alpha,i}$ diffeomorphically onto $V^{(n)}_{\alpha,i}$ such that the assertions of *Lemma 5.11* are fulfilled. As above, we get the estimate $\mathsf{area}\,(u_n(\sigma'_n(V_{\alpha,i})) \leqslant (N-1)\,\varepsilon$. This gives the inductive conclusion for *Subcase $3'''_b$)*.

Subcase $3''''_b$): The sequences $\left\{\frac{\rho_n}{r_n}\right\}$ and $\left\{\frac{R_n}{\rho_n}\right\}$ increases infinitely. Thus $\lim \frac{\tilde R_n}{R_n} = 1$. We consider the sequence of maps $\{v_n\}$. It is defined in the same way as in the previous subcase and has the same properties. In particular, $\{v_n\}$ converges to the map $v_\infty : S^2 \to X$ and there exists a sufficiently big $b > 0$ such that for $n >> 0$ we get the property (5.?16).

For $n >> 0$ define the coverings of $\mathcal{A}_0$ by 10 sets

$$
\begin{aligned}
&W^{(n)}_1 := \Delta'\backslash\overline{\Delta}'(0, \tfrac{\varrho}{2}), && W^{(n)}_2 := \Delta'(0,\varrho)\backslash\overline{\Delta}'(0,2R_n),\\
&W^{(n)}_3 := \Delta'(0,4R_n)\backslash\big(\overline{\Delta}'(0,\tfrac{\tilde R_n}{4})\cup\overline{\Delta}'(\tilde x_n,\tfrac{\tilde R_n}{4})\big),\\
&W^{(n)}_4 := \Delta'(0,\tfrac{\tilde R_n}{3})\cup\Delta'', && W^{(n)}_5 := \Delta'(\tilde x_n,\tfrac{\tilde R_n}{3})\backslash\overline{\Delta}'(\tilde x_n,2\rho_n),\\
&W^{(n)}_6 := \Delta'(\tilde x_n,4\rho_n)\backslash\big(\overline{\Delta}'(x_n,\tfrac{\rho_n}{4})\cup\overline{\Delta}'(\overline{x}_n,\tfrac{\rho_n}{4})\big),\\
&W^{(n)}_7 := \Delta'(x_n,\tfrac{\rho_n}{3})\backslash\overline{\Delta}'(x_n,br_n), && W^{(n)}_8 := \Delta'(x_n,2br_n),\\
&W^{(n)}_9 := \Delta'(\overline{x}_n,\tfrac{\rho_n}{3})\backslash\overline{\Delta}'(\overline{x}_n,br_n), && W^{(n)}_{10} := \Delta'(\overline{x}_n,2br_n).
\end{aligned}
$$

The rest "manipulations" with $W^{(n)}_i$ are the same as in the previous subcases. As result, we obtain the covering of V_α by sets $V_{\alpha,i} := \sigma^{-1}_{\alpha,n_0}(W^{(n_0)}_i)$ with an appropriate $n_0 >> 0$ and refined parametrizations $\sigma'_n : V_\alpha \to C_{\alpha,n}$, for which the assertions of *Lemma 5.11* are fulfilled. As above, we get the estimate $\mathsf{area}\,(u_n(\sigma'_n(V_{\alpha,i})) \leqslant (N-1)\,\varepsilon$. This gives the inductive conclusion for *Subcase $3''''_b$)*.

Case 4_b): V_α is a cylinder, such that conformal radii of $(V_\alpha, \sigma^*_n j_n)$ increase infinitely. We can simply repeat the contructions made in *Case 4)* from the proof of *Theorem 1.1*. An additional attention is needed to preserve τ-invariantness.

The proof of theorem can be now finished by induction. □

Remark. Here we give some explanation of the geometrical meaning of the constructions of the proof of *Theorem 5.9* and describe the picture of the bubbling. We restrict ourselves to *Case 3_b)* as the most complicated one, the constructions of other cases can be treated similarly. The reflection principle allows us to reduce the *Case 3_b)* to consideration of τ-invariant maps $u^d_n : \mathcal{A}_0 \to X$ from the standard node which are J_n-holomorphic on $\mathcal{A}^+_0$. The situation is most different from the situations of *Theorem 1.1* when the bubbling appears in the nodal point. In this case we must take into consideration not only parameters r_n describing the size of energy localization of the bubbled sphere, but also additional parameters R_n and ρ_n. These ones describe the position of the localization centers x_n w.r.t. the nodal point and the set of τ-invariant points of $\mathcal{A}_0$. Depending on the behavior of r_n ρ_n, and R_n we can have 4 different types of the bubbling and corrsponding *Subcases $3'_b$)–$3''''_b$)*.

In *Subcase $3'_b$)* the bubbling take place in the nodal point, so that the nodal point remains on the bubbled sphere (see region $W_3^{(n)}$ on Fig. 9). Furthermore, the bubbled sphere contains another one nodal point. This one appears in the limit of long cylinders $W_2^{(n)}$. Note $W_2^{(n)}$ can either strongly converge to a boundary node, or have additional bubblings.

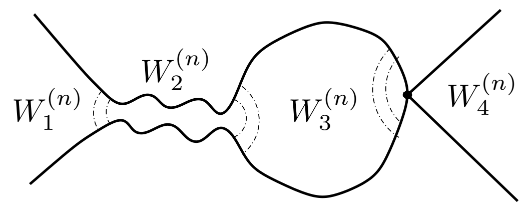

Fig. 9. Bubbling in *Subcase $3'_b$)*.

Turning back from "doubled" description by τ-invariant objects to the original maps $u_n : \mathcal{A}_0^+ \to X$ with totally real boundary condition we obtain the following picture. Since every covering piece $W_i^{(n)}$ is τ-invariant, for $\mathcal{A}_0^+$ we get the covering piece $W_i^{(n)+} := W_i^{(n)} \cap \mathcal{A}_0^+$. Thus we obtain a bubbled dics represented by $W_3^{(n)+}$ instead of the bubbled sphere represented by $W_3^{(n)}$, the sequence of long strips $W_2^{(n)+}$ instead of the sequence of long cylinder $W_2^{(n)+}$ and so on.

In *Subcase $3''_b$)* the bubbling happens at the boundary but away from the nodal point. In the limit we obtain 2 bubbled spheres. The first one is the limit of the rescaled maps v_n (region $W_6^{(n)}$ on the Fig. 10). The appearance of the second sphere can be explained as follows. The part of the node $\mathcal{A}_0$ between the first bubbled sphere $W_6^{(n)}$ and the "constant part" $W_1^{(n)}$ of the node is a long cylinder, represented by pieces $W_2^{(n)}$, $W_3^{(n)}$, and $W_5^{(n)}$.

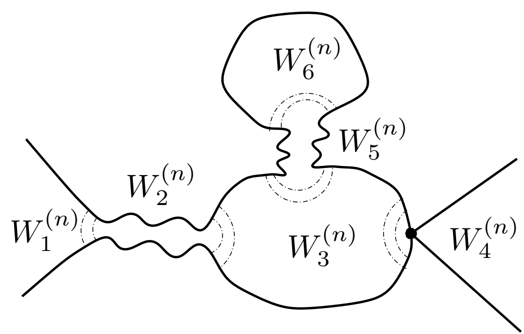

Fig. 10. Bubbling in *Subcase $3''_b$)*.

Howeverer, because of the presence of the nodal point (piece $W_4^{(n)}$ on the figure), this "part inbetween" is topologically not a cylinder (*i.e.* an annulus) but pants. Furthermore, the complex structures on the pants are not constant. To get pants with constant structure (piece $W_3^{(n)}$) we cut off the annuli $W_2^{(n)}$ and $W_5^{(n)}$. Since $\lim R_n = 0 = \lim \frac{r_n}{R_n}$, the conformal radii of these annuli increase infinitely. This shows that $W_2^{(n)}$ and $W_5^{(n)}$ are sequences of long cylinders and that the sequence $W_3^{(n)}$ defines in the limit a sphere with 3 nodal points.

As in *Subcase $3'_b$)* every covering piece $W_i^{(n)}$ is τ-invariant, whereas $W_i^{(n)+} := W_i^{(n)} \cap \mathcal{A}_0^+$ is the "half" of $W_n^{(n)}$. Thus for sequence of undoubled maps $u_n : \mathcal{A}_0^+ \to X$ we get the following picture of the bubbling . The limit contains 2 bubbled discs represented by $W_6^{(n)+}$ and $W_3^{(n)+}$, a boundary node $W_4^{(n)+}$, and possibly futher bubbled pieces which can appear in the limit of long strips $W_2^{(n)+}$ and $W_5^{(n)+}$. Note also that the action of the involution τ on the pants $W_3^{(n)}$ is described by Fig. 8 b).

In *Subcase $3'''_b$)* the bubbling takes place near, but not at the boundary. Indeed, since $\frac{\rho_n}{r_n} \longrightarrow \infty$, the bubbled sphere which appears as the limit of the sequence $\{v_n\}$ is not τ-invariant. To see this phenomenon we note that for any fixed $b > 0$ the

covering pieces $W_5^{(n)} = \Delta'(x_n, 2br_n)$ representing sufficient big part of this sphere lie in $\mathcal{A}_0^+$ for $n >> 0$. This implies that the sequence $v_n \circ \tau$ converges to another bubbled sphere, which is τ-symmetric to the first one and represented by $W_7^{(n)}$.

Another one bubbled sphere, represented by $W_3^{(n)}$, appears from pants between the first two spheres and the disc Δ'. Since $\{\frac{R_n}{\rho_n}\}$ remains bounded, the original nodal point remains on this latter sphere.

The corresponding bubbling picture for undoubled maps $u_n : \mathcal{A}_0^+ \to X$ is shown on on Fig. 11. The boundary of $\mathcal{A}_0^+$ is drawn by thick line. We obtain the bubbled sphere represented by $W_6^{(n)}$, the sequence of long cylinders $W_8^{(n)}$, the bubbled disc $W_3^{(n)+}$, and the sequence of long strips $W_2^{(n)+}$. Note that both sequences of long cylinders and long strips can yield further bubblings in the limit.

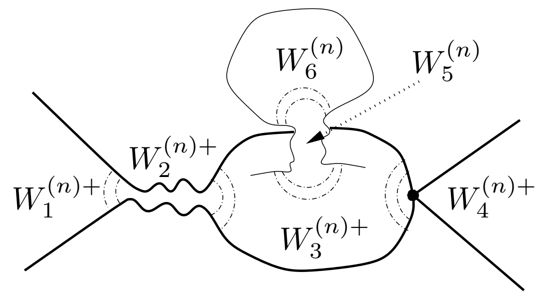

Fig. 11. Bubbling in *Subcase 3$_b'''$)*.

The bubbling picture in *Subcase 3$_b''''$)* is similar to the one of previous subcase, so we explain only the difference. It comes from the fact that the sequence $\{\frac{R_n}{\rho_n}\}$ is now unbounded, *i.e.* $\lim \frac{\rho_n}{R_n} = 0$. Informally speaking, this means that the long cylinder from *Subcase 3$_b'''$)* (piece $W_5^{(n)}$ on Fig. 11) moves to the boundary of the bubbled disc (piece $W_3^{(n)+}$ on Fig. 11). The precedure of additional rescaling divides every such dics into two new discs connected by a strip, pieces $W_3^{(n)+}$, $W_6^{(n)+}$, and $W_5^{(n)+}$ on Fig. 12 respectively. The infinite growth $\frac{R_n}{\rho_n} \to \infty$ means that $W_5^{(n)+}$ form a sequence of long strips.

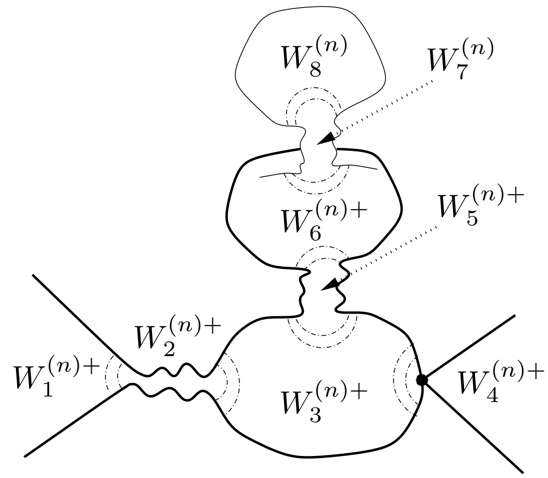

Fig. 16. Bubbling in *Subcase 3$_b''''$)*.

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